THREE INTERACTIONS OF HOLES IN TWO DIMENSIONAL DIMER SYSTEMS

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Abstract. Consider the unit triangular lattice in the plane with origin O, drawn so that one of the sets of lattice lines is vertical. Let l and l' denote respectively the vertical and horizontal lines that intersect O. Suppose the plane contains a pair of triangular holes of side length two, distributed symmetrically with respect to l and l', and oriented so that both holes point toward O. Unit rhombus tilings of three different regions of the plane are considered, namely: tilings of the entire plane; tilings of the half plane that lies to the left of l (where l is considered a free boundary, so unit rhombi are allowed to protrude halfway across it); and tilings of the half plane that lies just below the fixed boundary l'. Asymptotic expressions for the interactions of the triangular holes in these three different regions are obtained, providing further evidence for Ciucu's ongoing program that seeks to draw parallels between gaps in dimer systems on the hexagonal lattice and certain electrostatic phenomena.

1. Introduction

The study of interactions between holes (or gaps) in dimer systems was established by Fisher and Stephenson [15] in 1963 with their results concerning interactions of holes in dimer systems on the square lattice. Three specific types of interaction were presented: the interaction between a pair of dimer holes (that is, the interaction of two fixed dimers within a dimer system); the interaction between a pair of non-dimer holes (that is, the interaction between two fixed monomers in a dimer system); and the interaction of a dimer hole with a constrained boundary.

Kenyon [20] generalised the first of these results to an arbitrary number of dimer holes in dimer systems on both the square and hexagonal lattices. In [22] Kenyon, Okounkov and Sheffield further extended this work to general planar bipartite lattices. The interaction of a monomer with a straight line boundary in the square lattice is also due to Kenyon [21, Section 7.5], while the interaction of a family of holes with a constrained straight line boundary on the hexagonal lattice is tackled by Ciucu in [5, Theorem 2.2]. It is worth noting here that since the triangular lattice arises as the "dual" of the hexagonal lattice, dimer coverings of subgraphs of the hexagonal lattice are often much more easily thought of in terms of unit rhombus tilings of sub-regions of the triangular lattice in the plane (so monomers correspond to unit triangles and

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dimers correspond to unit rhombi). A dimer covering of the entire hexagonal lattice then corresponds to a tiling of the entire plane by unit rhombi (sometimes referred to as a sea of unit rhombi).

Ciucu has also studied various types of interactions between non-dimer holes in dimer systems on the hexagonal lattice (equivalently, interactions between triangular holes within a sea of unit rhombi [3][4][5][6][8]), thereby establishing close analogies to certain electrostatic phenomena. Ciucu conjectures that interactions between holes in dimer systems on the hexagonal lattice are governed by the laws of two dimensional electrostatics (that is, Coulomb's laws). More specifically, the conjecture states that by taking the exponential of the negative of the electrostatic energy of the two dimensional system of physical charges obtained by considering each hole as a point charge of size and magnitude specified by a statistic[1] q, one may recover (up to a multiplicative constant) asymptotic expressions for the dimer-mediated interactions of holes on the hexagonal lattice. This conjecture remains wide open, though it has been shown to hold for two fairly general families of holes.

In light of this, further types of interaction have been studied. Recently in [13], Ciucu and Krattenthaler determined the interaction between a left-pointing triangular hole of side length two and a vertical free boundary that borders a sea of unit rhombi on the left half of the plane. It would appear that this result (which inspired the present paper) is the first treatment of such an interaction in the literature. Theorem 2.2 below is a direct analogy of the interaction presented in [13] in the sense that here the hole has been "flipped" and instead points toward the free boundary. Curiously, Theorem 2.2 agrees with that of [13] only up to a multiplicative constant.

As well as Theorem 2.2, this paper establishes two other types of interaction between a pair of triangular holes of side length two that point towards each other in dimer systems (that is, a pair of triangles oriented so that one left-pointing triangle lies directly to the right of one right-pointing triangle). These interactions are: the interaction between such a pair of holes that lie within a sea of unit rhombi (Theorem 2.1); and the interaction between such a pair of holes that lie on a fixed boundary of a sea of unit rhombi (Theorem 2.3). All three of these results agree with the original conjecture of Ciucu and so serve as yet more evidence that interactions between holes in dimer systems are governed by the laws of two dimensional electrostatics. A brief outline of the paper follows.

Section 2 discusses the correspondence between dimer systems on the hexagonal lattice and rhombus tilings of hexagons on the triangular lattice, and also states precisely the family of holey hexagons (that is, hexagons containing triangular holes) that are of particular interest. Certain symmetry classes of rhombus tilings of a holey hexagon correspond precisely to rhombus tilings of smaller sub-regions of the same hexagon. The three different types of interaction that are the focus of this work arise from considering tilings of these sub-regions as they become infinitely large. These sub-regions are defined in Section 2, along with the corresponding interactions (or correlation functions). Asymptotic expressions for these interactions are also stated here (Theorem 2.1, Theorem 2.2, and Theorem 2.3), which together comprise the main results. Proposition 2.4–

[1]For a triangular hole t on the unit triangular lattice, $q(t)$ is defined to be the signed difference between the number of left and right-pointing unit triangles that comprise t.

a factorisation result that is frequently used throughout this paper– is also stated in this section.

Section 3 briefly discusses the relationship between rhombus tilings of hexagons and plane partitions. This is followed by four exact enumeration formulas that count tilings of sub-regions of holey hexagons (Theorem 3.1, Theorem 3.2, Theorem 3.3, and Theorem 3.4). Proofs of these theorems follow in Sections 4 and 5.

In Section 4 the well-known bijection between rhombus tilings and non-intersecting lattice paths is combined with existing results due to Stembridge [31] and Lindström-Gessel-Viennot [16] in order to express the number of tilings of different regions as either a Pfaffian (Proposition 4.3) or a determinant (Proposition 4.4, Proposition 4.5). In Section 5 the matrices defined in Proposition 4.3 are manipulated using standard row and column operations in such a way that the Pfaffian of each matrix may be expressed as a determinant of a much smaller matrix. The unique LU-decompositions of these smaller matrices, along with those from Proposition 4.4 and Proposition 4.5, are then stated and proved (Theorem 5.2, Lemma 5.3, Theorem 5.4, and Theorem 5.5), from which the exact enumeration formulas in Section 3 immediately follow.

Section 6 contains proofs of the main results. These follow from inserting the enumeration formulas from Section 3 into the correlation functions defined in Section 2. Asymptotic expressions for the three types of interaction are then derived by studying these correlation functions as the distance between the holes (or the distance between the hole and the free boundary) becomes very large.

2. Set-Up and Results

Consider the hexagonal lattice $\mathscr{H}$ in terms of its dual $\mathscr{T}$ (that is, the planar triangular lattice consisting of unit equilateral triangles drawn so that one of the sets of lattice lines is vertical). Then monomers on $\mathscr{H}$ correspond to unit triangles on $\mathscr{T}$ and dimers on $\mathscr{H}$ correspond to unit rhombi on $\mathscr{T}$ (a dimer on $\mathscr{H}$ is equivalent to joining two unit triangles that share exactly one edge on $\mathscr{T}$). It follows that dimer coverings of regions of $\mathscr{H}$ that contain gaps correspond exactly to rhombus tilings of regions of $\mathscr{T}$ that contain triangular holes.

Fix some origin O in $\mathscr{T}$ and denote by $H_{a,b}$ the hexagonal region centred at O with side lengths a, b, a, a, b, a (going clockwise from the southwest edge). Suppose T is a set of fixed unit triangles contained within the interior of $H_{a,b}$, where the distances between the triangles are parametrised by k. Denote by $H_{a,b} \setminus T$ the hexagonal region $H_{a,b}$ with the set of triangles T removed from its interior (such regions will often be referred to as *holey hexagons*, a term coined by Propp in [28]). Suppose further that $\xi > 0$ is real such that $b \sim \xi a$ is integral. Then the *interaction* (otherwise known as the *correlation function*, or simply the *correlation*) between the holes in T is defined to be

$$\omega(k;\xi) = \lim_{a\to\infty} \frac{M(H_{a,b} \setminus T)}{M(H_{a,b})},$$

where $M(R)$ denotes the number of rhombus tilings of the region R. Different types of interactions may be obtained by replacing $H_{a,b} \setminus T$ and $H_{a,b}$ with specific sub-regions of $H_{a,b} \setminus T$ (and $H_{a,b}$ respectively) in the above formula. Note that in order for a rhombus tiling of $H_{a,b}$ to exist (and hence for the above definition to make sense), it must be the

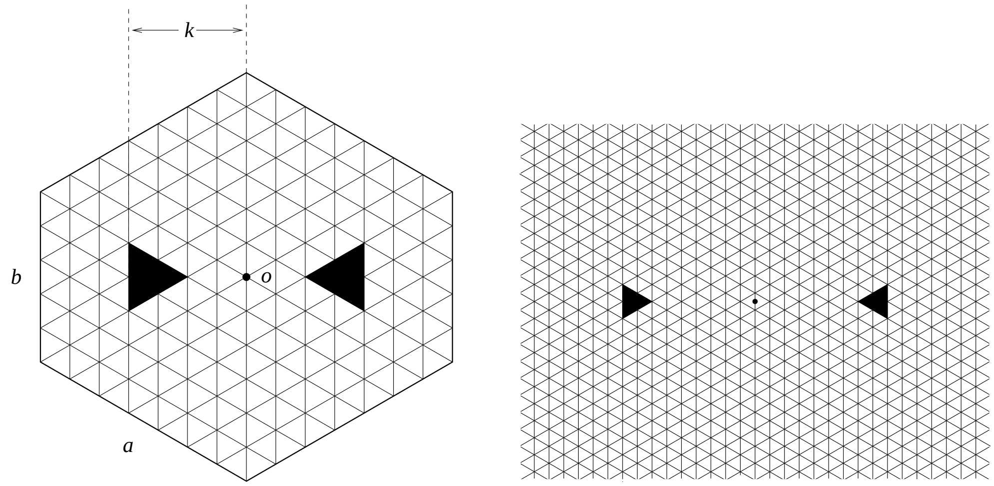

FIGURE 1. Left: the holey hexagon $H_{7,5}\setminus(\triangleright_4\cup\triangleleft_4)$. Right: a small section of the plane of unit triangles given by $H_{a,b}\setminus(\triangleright_9\cup\triangleleft_9)$ as a tends to infinity.

case that $\sum_{t\in T} q(t) = 0$, where $q(t)$ is the statistic defined in Section 1. The triangular holes and sub-regions of particular interest are defined below.

For some non-negative integer k, let $\triangleright_k$ denote the right-pointing equilateral triangle of side length two at lattice distance k to the left of the origin such that the centre of its vertical side lies on the horizontal symmetry axis of $H_{a,b}$. Define $\triangleleft_k$ analogously, that is, $\triangleleft_k$ denotes the left-pointing triangle of side length two with its vertical edge at lattice distance k to the right of the origin. Then $H_{a,b}\setminus(\triangleright_k\cup\triangleleft_k)$ denotes the hexagonal region $H_{a,b}$ from which two inward pointing triangular holes of side length two have been removed that are symmetrically distributed with respect to the vertical and horizontal symmetry axes of $H_{a,b}$. Figure 1 (left) shows such a region for $a = 7$, $b = 5$ and $k = 4$.

Remark 2.1. Under this construction the parity of a and b determine the parity of k, that is, if a and b are of the same parity then k is even, otherwise k is odd.

Consider $H_{a,b}\setminus(\triangleright_k\cup\triangleleft_k)$ as a is sent to infinity. It should be clear that this gives the entire plane of unit triangles containing two triangular holes with their vertical sides at lattice distance $2k$ apart, a portion of which may be seen in Figure 1 (right). Hence the correlation function

$$\omega_H(k;\xi) = \lim_{a\to\infty}\frac{M(H_{a,b}\setminus(\triangleright_k\cup\triangleleft_k))}{M(H_{a,b})}$$

is the interaction (in terms of k) between a pair of triangular holes of side length 2 oriented so that they point directly at each other within a sea of unit rhombi.

Theorem 2.1. *As k becomes very large the interaction defined above, $\omega_H(k;\xi)$, is asymptotically*

$$\omega_H(k;\xi) \sim \left(\frac{1}{2k\pi}\left(\frac{\xi+1}{2}\right)^{2k-2}\right)^2.$$

Denote by $V_{a,b}\setminus\triangleright_k$ the left half of $H_{a,b}\setminus(\triangleright_k\cup\triangleleft_k)$ whose rightmost boundary is the vertical line l that intersects the origin. If l is a free boundary (that is, unit rhombi are

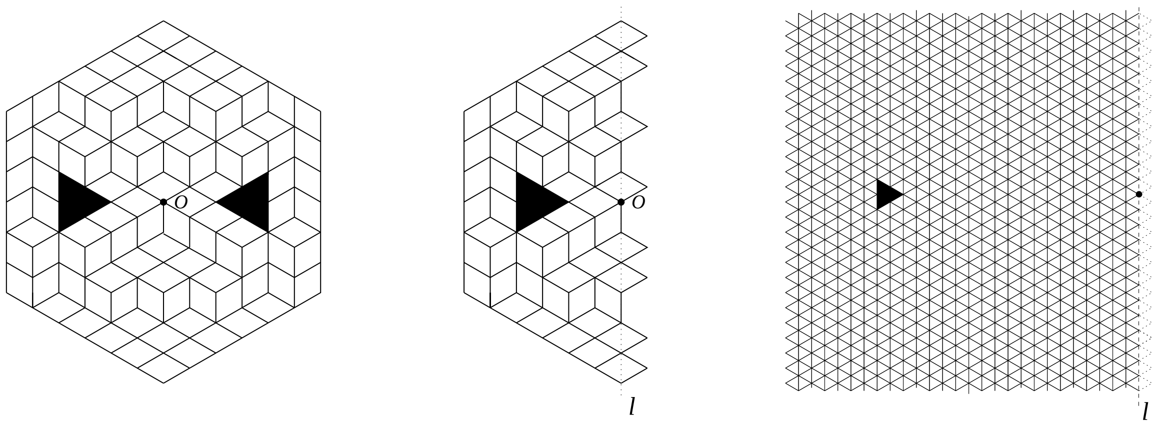

FIGURE 2. Left: a vertically symmetric rhombus tiling of $H_{6,6} \setminus (\triangleright_4 \cup \triangleleft_4)$. Centre: the corresponding tiling of $V_{6,6} \setminus \triangleright_4$. Right: a small section of the left half of the plane of unit triangles given by $V_{a,b} \setminus \triangleright_{20}$ as a tends to infinity.

permitted to protrude across l), then the number of rhombus tilings of $V_{a,b} \setminus \triangleright_k$ is clearly equal to the number of vertically symmetric tilings of $H_{a,b} \setminus (\triangleright_k \cup \triangleleft_k)$. Figure 2 (left) shows such a tiling for $H_{6,6} \setminus (\triangleright_4 \cup \triangleleft_4)$ along with the corresponding tiling of $V_{6,6} \setminus \triangleright_4$ (centre).

As a tends to infinity, the region $V_{a,b} \setminus \triangleright_k$ gives the left half plane of unit triangles, constrained on the right by l, and containing a right-pointing triangular hole with its vertical side at lattice distance k to the left of l (see Figure 2, right). Hence the correlation function

$$\omega_V(k;\xi) = \lim_{a\to\infty} \frac{M(V_{a,b} \setminus \triangleright_k)}{M(V_{a,b})}$$

gives the interaction (in terms of k) between a right-pointing triangle of side length two and a vertical free boundary that borders a sea of unit rhombi that tile the left half plane.

Theorem 2.2. *As k becomes very large the interaction defined above, $\omega_V(k;\xi)$, is asymptotically*

$$\omega_V(k;\xi) \sim \frac{\sqrt{\xi(\xi+2)}}{2\pi k} \left(\frac{(\xi+1)}{2}\right)^{2(k-1)}.$$

Consider now the two regions obtained by cutting $H_{a,b} \setminus (\triangleright_k \cup \triangleleft_k)$ along the zig-zag line that lies just below its horizontal symmetry axis. Denote the upper and lower portions of $H_{a,b} \setminus (\triangleright_k \cup \triangleleft_k)$ with $H^+_{a,b} \setminus (\triangleright_k \cup \triangleleft_k)$ and $H^-_{a,b} \setminus (\triangleright_k \cup \triangleleft_k)$ respectively. Figure 3 (left) shows the upper and lower regions of $H_{7,8} \setminus (\triangleright_5 \cup \triangleleft_5)$. If b is even then rhombus tilings of $H^-_{a,b} \setminus (\triangleright_k \cup \triangleleft_k)$ correspond to horizontally symmetric tilings of the entire region $H_{a,b} \setminus (\triangleright_k \cup \triangleleft_k)$.

Now consider $H^-_{a,b} \setminus (\triangleright_k \cup \triangleleft_k)$ as a is sent to infinity. Clearly one obtains the lower half plane of unit triangles, bounded above by a fixed boundary on which lie two triangular holes with vertical sides at lattice distance $2k$ apart– see Figure 3 (right).

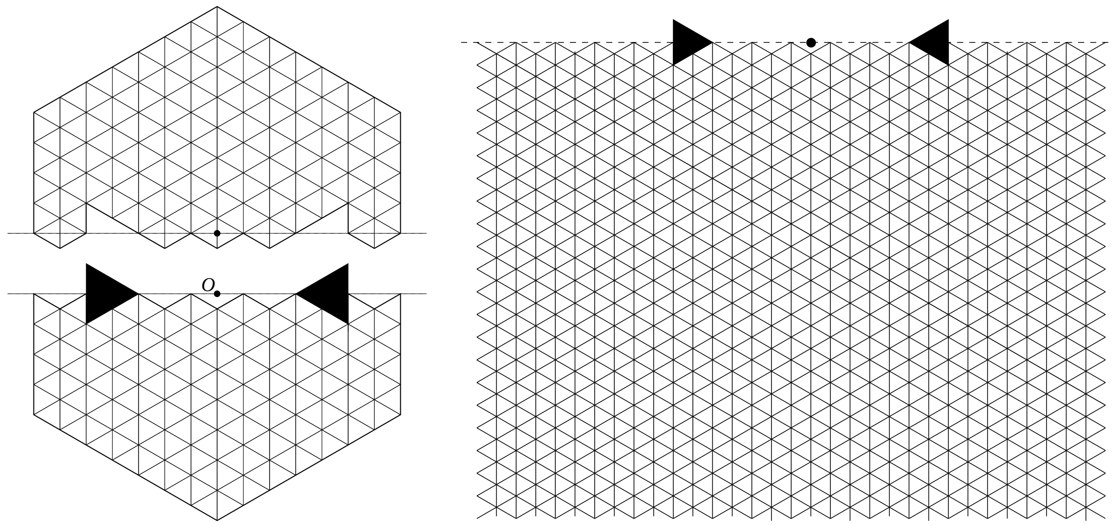

FIGURE 3. Left: The regions $H^+_{7,8} \setminus (\triangleright_4 \cup \triangleleft_4)$ (upper) and $H^-_{7,8} \setminus (\triangleright_4 \cup \triangleleft_4)$ (lower). Right: a small section of the lower half of the plane of unit triangles given by $H^-_{a,b} \setminus (\triangleright_7 \cup \triangleleft_7)$ as a tends to infinity.

The correlation function

$$\omega_{H^-}(k;\xi) = \lim_{a\to\infty} \frac{M(H^-_{a,b} \setminus (\triangleright_k \cup \triangleleft_k))}{M(H^-_{a,b})}$$

then gives the interaction (in terms of k) of a pair of triangular holes of side length two that point directly towards each other and are positioned along the fixed upper boundary that borders unit rhombus tilings of the lower half plane.

Theorem 2.3. *As k becomes very large the correlation function defined above, $\omega_{H^-}(k;\xi)$, is asymptotically equivalent to that of $\omega_V(k;\xi)$ from Theorem 2.2.*

Observe that for $\xi \neq 1$, Theorem 2.1 and Theorem 2.2 give distorted dimer statistics. Indeed, these expressions blow up or vanish exponentially for $\xi > 1$ and $\xi < 1$ respectively. So when considering the three types of interaction detailed previously, it makes sense to consider only regular hexagons, that is, hexagons with all sides the same length. In light of this the above asymptotic expressions may be re-packaged in the following way:

$$\omega_V(k;1) \sim \frac{3}{4\pi \mathrm{d}(\triangleright_k, O)} \tag{2.1}$$

$$\omega_H(k;1) \sim \left(\frac{\sqrt{3}}{\pi \mathrm{d}(\triangleright_k, \triangleleft_k)}\right)^2 \tag{2.2}$$

where $\mathrm{d}(a,b)$ denotes the Euclidean distance between a and b.

The first result above is analogous to that of Ciucu and Krattenthaler [13], which presents an asymptotic result for the correlation of a triangular hole that has been "flipped" and instead points away from the vertical free boundary l. Within an infinitely large sea of rhombi one could be forgiven for thinking that the orientation of such a hole would have no effect on the interaction between the hole and the free boundary,

yet this result shows that such an interaction differs by a factor of 3. Any intuitive reason as to why this phenomenon should occur remains a complete mystery.

Both of these results are entirely in keeping with the original conjecture of Ciucu [4, Conjecture 1]. The asymptotic expression for $\omega_H(k;\xi)$ may be obtained (up to a multiplicative constant) by considering the exponential of the negative of the electrostatic energy of the system of physical charges obtained by viewing the triangular holes as point charges, where one has signed magnitude $+2$ and the other -2 according to the definition of the statistic q in Section 1. In a similar way the asymptotic expression for $\omega_V(k;\xi)$ shows that a triangular gap is attracted to a vertical free boundary l in precisely the same way that an electrical charge is attracted to a straight line conductor when placed near it. Hence (2.1) and (2.2) above further support the hypothesis that interactions of gaps in the hexagonal lattice are governed by Coulomb's laws for two dimensional electrostatics.

This section concludes with an important proposition concerning the enumeration of tilings of the entire region $H_{a,b} \setminus (\triangleright_k \cup \triangleleft_k)$. It follows from the Matchings Factorization Theorem [7] by employing arguments that are almost identical to those of Ciucu and Krattenthaler [12].

Proposition 2.4. *The total number of tilings of the region $H_{a,b} \setminus (\triangleright_k \cup \triangleleft_k)$ has the factorisation*

$$M(H_{a,b} \setminus (\triangleright_k \cup \triangleleft_k)) = M(H^-_{a,b} \setminus (\triangleright_k \cup \triangleleft_k)) \cdot M_w(H^+_{a,b} \setminus (\triangleright_k \cup \triangleleft_k)),$$

where $M(H)$ denotes the number of tilings of the region H and $M_w(H^+_{a,b} \setminus (\triangleright_k \cup \triangleleft_k))$ denotes the weighted count of the region $H^+_{a,b} \setminus (\triangleright_k \cup \triangleleft_k)$ where the unit rhombi that lie along its bottom edge carry a weight of 2*.*

Remark 2.2. From now on, denote by $\overline{H}^+_{a,b} \setminus (\triangleright_k \cup \triangleleft_k)$ the region $H^+_{a,b} \setminus (\triangleright_k \cup \triangleleft_k)$ with the added condition that the unit rhombi lying along its bottom edge have a weight of 2, so that $M_w(H^+_{a,b} \setminus (\triangleright_k \cup \triangleleft_k) = M(\overline{H}^+_{a,b} \setminus (\triangleright_k \cup \triangleleft_k))$.

This proposition implies that rhombus tilings of $H_{a,b} \setminus (\triangleright_k \cup \triangleleft_k)$ may be enumerated by counting separately the number of tilings of $\overline{H}^+_{a,b} \setminus (\triangleright_k \cup \triangleleft_k)$ and $H^-_{a,b} \setminus (\triangleright_k \cup \triangleleft_k)$ and taking their product. If b is an even integer then tilings of the latter region correspond to horizontally symmetric tilings of the entire hexagon $H_{a,b} \setminus (\triangleright_k \cup \triangleleft_k)$. By enumerating tilings of $H_{a,b} \setminus (\triangleright_k \cup \triangleleft_k)$ in terms of this factorisation formula one obtains, for free, an enumeration formula for horizontally symmetric tilings of $H_{a,b} \setminus (\triangleright_k \cup \triangleleft_k)$.

A similar correlation function involving $\overline{H}^+_{a,b} \setminus (\triangleright_k \cup \triangleleft_k)$ may also be defined, namely

$$\omega_{H^+}(k;\xi) = \lim_{a\to\infty} \frac{M(\overline{H}^+_{a,b} \setminus (\triangleright_k \cup \triangleleft_k))}{M(\overline{H}^+_{a,b})},$$

which gives the interaction between a pair of triangular holes of side length two that lie on the fixed weighted boundary of a sea of unit rhombi. According to [12], the number of rhombus tilings of $\overline{H}^+_{a,b}$ is equal to the number of tilings of $V_{a,b}$ and so this correlation function becomes

$$\omega_{H^+}(k;\xi) = \lim_{a\to\infty} \frac{M(\overline{H}^+_{a,b} \setminus (\triangleright_k \cup \triangleleft_k))}{M(V_{a,b})}.$$

Theorem 2.2 above is obtained by directly analysing $\omega_V(k;\xi)$ as k becomes large, whereas Theorem 2.1 follows from considering the product of $\omega_{H^+}(k;\xi)$ and $\omega_{H^-}(k;\xi)$ asymptotically. In order to do so, it is necessary to first find exact enumeration formulas for $M(V_{a,b} \setminus \triangleright_k)$, $M(H^-_{a,b} \setminus (\triangleright_k \cup \triangleleft_k))$, and $M(\overline{H}^+_{a,b} \setminus (\triangleright_k \cup \triangleleft_k))$. Sections 3, 4 and 5 are concerned with stating and proving these exact formulas, while the proofs of Theorems 2.1, 2.2 and 2.3 may be found in Section 6.

3. Some Exact Enumeration Formulas

The enumeration of rhombus tilings (in various forms and guises) has a fairly long history beginning with MacMahon [25] in the early 20th century. Although at the time MacMahon was interested in counting the number of plane partitions whose Young tableaux fit inside an $a \times b \times c$ box, there is a straightforward bijection between the three dimensional representation of these objects (as unit cubes stacked in the corner of an $a \times b \times c$ box) and two dimensional unit rhombus tilings of $H_{a,b,c}$, the hexagon with side lengths a, b, c, a, b, c (going clockwise from the southwest edge). Under this same bijection symmetric plane partitions whose Young tableaux fit inside an $a \times a \times b$ box correspond to vertically symmetric rhombus tilings of $H_{a,b}$. Similarly, transpose-complementary plane partitions whose Young tableaux fit inside an $a \times a \times 2b$ box correspond to horizontally symmetric tilings of $H_{a,2b}$.

In [25, p. 270] MacMahon conjectured a formula for the weighted enumeration of symmetric plane partitions of height at most b whose Young Tableaux fit inside an $a \times a \times b$ box. Under some specialisation, MacMahon's conjectured formula counts vertically symmetric rhombus tilings of $H_{a,b}$. Andrews managed to prove MacMahon's conjecture in full in [1], and hence established the (ordinary) enumeration of symmetric plane partitions. As far as ordinary enumeration goes, however, Andrews was not the first to prove the corresponding result. It was Gordon who, around 1970 (though published much later in [17]), proved the so-called Bender-Knuth conjecture [2], which for a special case gives the ordinary enumeration of symmetric plane partitions. Since then further refinements and alternative proofs have been found by Macdonald [24, pp. 83–85], Proctor [27, Prop. 7.3], Fischer [14] and Krattenthaler [23, Theorem 13]. This theorem states (in an equivalent form) that the number of vertically symmetric rhombus tilings of $H_{a,b}$ (that is, the number of rhombus tilings of the region $V_{a,b}$) is

$$ST(a,b) = \prod_{i=1}^{a} \frac{2i+b-1}{2i-1} \prod_{1 \le i < j \le a} \frac{i+j+b-1}{i+j-1}.$$

A similar result holds for rhombus tilings of $V_{a,b} \setminus \triangleright_k$.

Theorem 3.1. *Suppose k is a non-negative integer.*

(i) The number of vertically symmetric tilings of $H_{n,2m} \setminus (\triangleright_k \cup \triangleleft_k)$ is

$$\left[\sum_{s=1}^{m} B_{n,k}(s) \cdot D_{n,k}(s)\right] \times ST(n,2m),$$

where

$$B_{n,k}(s) = \frac{(-1)^{s+1}(-k+n+1)!(n+s-1)!(n+2s-1)!\left(\frac{k}{2}+\frac{n}{2}+s-2\right)!}{(s-1)!\left(\frac{n}{2}-\frac{k}{2}\right)!\left(\frac{k}{2}+\frac{n}{2}-1\right)!(2n+2s-1)!\left(-\frac{k}{2}+\frac{n}{2}+s\right)!},$$

$$D_{n,k}(s) = \frac{(-1)^{s+1}(2s-2)!(n-k)!(n+s-1)!\left(\frac{k}{2}+\frac{n}{2}+s-2\right)!}{(s-1)!\left(\frac{n}{2}-\frac{k}{2}\right)!\left(\frac{k}{2}+\frac{n}{2}-1\right)!(n+2s-2)!\left(-\frac{k}{2}+\frac{n}{2}+s\right)!} + \frac{2(-1)^{s+1}(2s-2)!(n-k)!(n+s)!\left(\frac{k}{2}+\frac{n}{2}+s-2\right)!}{(s-2)!\left(\frac{n}{2}-\frac{k}{2}\right)!\left(\frac{k}{2}+\frac{n}{2}\right)!(n+2s-2)!\left(-\frac{k}{2}+\frac{n}{2}+s\right)!}.$$

(ii) The number of vertically symmetric tilings $H_{n,2m-1} \setminus (\triangleright_k \cup \triangleleft_k)$ *is*

$$ST(n,2m-1) \times \left[\sum_{s=1}^{m-1}\left(E^*_{n,k}(s) - \frac{D^*_n(s)E^*_{n,k}(m)}{D^*_n(m)}\right)\cdot B^*_{n,k}(s)\right]$$

where

$$B^*_{n,k}(j) = \frac{2(-1)^{j+1}(j+n-1)!(2j+n-1)!(n-k)!\left(\frac{1}{2}(2j+k+n-5)\right)!}{(j-1)!(2j+2n-1)!\left(\frac{1}{2}(-k+n-1)\right)!\left(\frac{1}{2}(k+n-3)\right)!\left(\frac{1}{2}(2j-k+n+1)\right)!} + \frac{4(-1)^{j+1}(j+n)!(2j+n-1)!(n-k)!\left(\frac{1}{2}(2j+k+n-5)\right)!}{(j-2)!(2j+2n-1)!\left(\frac{1}{2}(-k+n-1)\right)!\left(\frac{1}{2}(k+n-1)\right)!\left(\frac{1}{2}(2j-k+n+1)\right)!}$$

$$D^*_n(i) = \frac{2^n(2i-2)!(i+n-1)!}{(i-1)!(2i+n-2)!},$$

$$E^*_{n,k}(i) = \frac{(-1)^{i+1}(2i-2)!(i+n-1)!(n-k)!\left(\frac{1}{2}(2i+k+n-3)\right)!}{(i-1)!(2i+n-2)!\left(\frac{1}{2}(-k+n-1)\right)!\left(\frac{1}{2}(k+n-1)\right)!\left(\frac{1}{2}(2i-k+n-1)\right)!}.$$

The following formula, due to Proctor [27], counts the number of transpose-complementary plane partitions that fit inside an $a \times a \times 2b$ box:

$$TC(a,2b) = \binom{a+b-1}{a-1}\prod_{i=1}^{a-2}\prod_{j=i}^{a-2}\frac{2b+i+j+1}{i+j+1}.$$

Since transpose complementary plane partitions correspond to horizontally symmetric tilings of $H_{a,b}$, the above formula counts precisely the number of rhombus tilings of the region $H^-_{a,2b}$. Again, a similar result holds for rhombus tilings of $H^-_{a,2b} \setminus (\triangleright_k \cup \triangleleft_k)$.

Theorem 3.2. *The number of horizontally symmetric tilings of* $H_{n,2m} \setminus (\triangleright_k \cup \triangleleft_k)$ *(equivalently, the number of tilings of the region* $H^-_{n-1,2m+1} \setminus (\triangleright_k \cup \triangleleft_k)$*) is given by*

$$\left[\sum_{s=1}^{m} B'_{n,k}(s)\cdot D'_{n,k}(s)\right]\cdot TC(n,2m),$$

where

$$B'_{n,k}(s) = \frac{(-1)^{s+1}(n-k)!(n+s-2)!(n+2s-1)!\left(\frac{1}{2}(k+n+2s-4)\right)!}{2(s-1)!\frac{n-k}{2}!\left(\frac{1}{2}(k+n-2)\right)!(2n+2s-3)!\left(-\frac{k}{2}+\frac{n}{2}+s\right)!},$$

$$D'_{n,k}(s) = \frac{(-1)^{s+1}(2s)!(n-k)!(n+s-1)!\left(\frac{1}{2}(k+n+2s-4)\right)!}{2(s!)\frac{n-k}{2}!\left(\frac{1}{2}(k+n-2)\right)!(n+2s-2)!\left(-\frac{k}{2}+\frac{n}{2}+s\right)!}.$$

Remark 3.1. Observe that if b is odd there exist no horizontally symmetric tilings of the region $H_{a,b}$.

The following enumerative result counts the number of tilings of the weighted region $\overline{H}^{+}_{a,b} \setminus (\triangleright_k \cup \triangleleft_k)$ defined in the previous section.

Theorem 3.3. *The number of rhombus tilings of $\overline{H}^{+}_{n,2m} \setminus (\triangleright_k \cup \triangleleft_k)$, the upper half of the region $H_{n,2m} \setminus (\triangleright_k \cup \triangleleft_k)$ containing lozenges of weight 2 lying along the horizontal symmetry axis is*

$$\left[\sum_{s=1}^{m} B_{n,k}(s) \cdot E_{n,k}(s)\right] \cdot ST(n, 2m),$$

where $B_{n,k}(s)$ is defined as in Theorem 3.1 and

$$E_{n,k}(s) = \frac{(-1)^{s+1}(2s-2)!(-k+n+1)!(n+s-1)!\left(\frac{k}{2}+\frac{n}{2}+s-2\right)!}{(s-1)!\left(\frac{n}{2}-\frac{k}{2}\right)!\left(\frac{k}{2}+\frac{n}{2}-1\right)!(n+2s-2)!\left(-\frac{k}{2}+\frac{n}{2}+s\right)!}.$$

Similarly, the number of tilings of $\overline{H}^{+}_{n+1,2m-1} \setminus (\triangleright_k \cup \triangleleft_k)$ is equal to $2 \cdot M(\overline{H}^{+}_{n,2m} \setminus (\triangleright_k \cup \triangleleft_k))$.

Remark 3.2. By re-writing the above result in terms of Theorem 2.2 and combining it with Theorem 3.2 via Proposition 2.4, one may construct a factorisation for $H_{a,b} \setminus (\triangleright_k \cup \triangleleft_k)$ that is in a sense similar to that of Ciucu and Krattenthaler [12]. Indeed, it was the observation of this factorisation that lead to the results presented in the current work.

MacMahon proved in [25] that the total number of plane partitions of an $a \times b \times c$ box (equivalently, the number of rhombus tilings of the hexagon $H_{a,b,c}$) is given by the formula

$$T(a, b, c) = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}.$$

Combining Theorem 3.2 with Theorem 3.3 via Proposition 2.4 gives a similar formula for enumerating tilings of $H_{a,b} \setminus (\triangleright_k \cup \triangleleft_k)$.

Theorem 3.4. *The total number of tilings of the region $H_{n,2m} \setminus (\triangleright_k \cup \triangleleft_k)$ is*

$$\left[\sum_{s=1}^{m} B'_{n,k}(s) \cdot D'_{n,k}(s)\right] \times \left[\sum_{t=1}^{m} B_{n,k}(t) \cdot E_{n,k}(t)\right] \times T(n, 2m, n),$$

whilst the number of tilings of $H_{n,2m-1} \setminus (\triangleright_k \cup \triangleleft_k)$ is

$$\left[\sum_{s=1}^{m} B_{n-1,k}(s) \cdot E_{n-1,k}(s)\right] \times \left[\sum_{t=1}^{m-1} B'_{n+1,k}(t) \cdot D'_{n+1,k}(t)\right] \times T(n, 2m-1, n).$$

Remark 3.3. If $k = n$, then the first formula in the above theorem gives the number of rhombus tilings of the hexagon $H_{n,2m}$ containing two unit triangular dents in each vertical side, positioned symmetrically with respect to the horizontal symmetry axis. This is a special case of the more general class of rhombus tilings considered by Ciucu and Fischer in [11].

The following two sections are dedicated to proving Theorem 3.1, Theorem 3.2 and Theorem 3.3 by translating sets of rhombus tilings into families of non-intersecting paths (Section 4). The enumeration of these families of paths can be expressed as the

determinant (or Pfaffian) of certain matrices. These determinants and Pfaffians are then evaluated in Section 5.

4. Non-intersecting Lattice Paths

The aim of this section is to express the number of rhombus tilings of $V_{a,b} \setminus \triangleright_k$, $\overline{H}^+_{a,b} \setminus (\triangleright_k \cup \triangleleft_k)$ and $H^-_{a,b} \setminus (\triangleright_k \cup \triangleleft_k)$ as determinants (or Pfaffians) of particular matrices.

Consider first the region $V_{a,b} \setminus \triangleright_k$ described in Section 2. Tilings of this region may be represented by unique families of lattice paths across dimers (see Figure 4, left), which may in turn be translated into unique families of non-intersecting lattice paths consisting of north and east unit steps on the integer lattice with origin O' (see Figure 4, right). Counting rhombus tilings of $V_{a,b} \setminus \triangleright_k$ is therefore equivalent to counting the number of families $(P_1, P_2, ..., P_b)$ of non-intersecting lattice paths between a set of starting points $A = \{(-s, s) : 1 \le s \le b\}$ and a set of end points $I = \{(x, y) \in \mathbb{Z}^2 : x + y = a\} \cup \{t^-, t^+\}$, with the requirement that $t^+ = ((a - b - k)/2 - 1, (a + b - k)/2 + 1)$ and $t^- = ((a - b - k)/2, (a + b - k)/2)$ must be included as end points. Figure 4 shows the unique family of lattice paths corresponding to the tiling found in Figure 2.

By the well-known theorem of Stembridge [31], such families of non-intersecting paths may be expressed in terms of a *Pfaffian*, defined below.

Definition 4.1. *The Pfaffian of a $2n \times 2n$ skew-symmetric matrix A is*

$$\mathrm{Pf}(A) = \sum_{\pi \in \mathcal{M}_{2n}} \mathrm{sgn}(\pi) \prod_{i<j \text{ matched in } \pi} (A)_{i,j},$$

where $\mathcal{M}_{2n}$ denotes the set of all perfect matchings on the set of vertices $\{1, 2, \ldots, 2n\}$ and $sgn(\pi) = (-1)^{cr(\pi)}$, where $cr(\pi)$ is the number of crossings of π.

Remark 4.1. It is a well known fact that

$$\mathrm{Pf}(A)^2 = \det(A).$$

Theorem 4.2 (Stembridge)**.** *Let $V = \{v_1, ..., v_r\}$, $U = \{u_1, ..., u_p\}$ and $E = \{e_1, ..., e_q\}$ be finite sets of lattice points in the integer lattice $\mathbb{Z}^2$ and assume that $p + r$ is even. Denote by $\mathscr{P}(A \to B)$ the total number of families of non-intersecting paths from the set A to the set B, whilst $P(a \to b)$ denotes the number of paths starting at point a and ending at point b. If E is totally ordered in such a way that U is D-compatible*[2] *with $V \oplus E$ (disjoint union), then the total number of non-intersecting paths starting at U and ending at $v \oplus E$ is*

$$\mathscr{P}(U \to v \oplus E) = \mathrm{Pf}\begin{pmatrix} Q & Y \\ -Y^t & 0 \end{pmatrix},$$

where $Q = (Q_{i,j})_{1 \le i,j \le p}$ is given by

$$Q_{i,j} = \sum_{1 \le s < t \le q} (P(u_i \to e_s)P(u_j \to e_t) - P(u_i \to e_t)P(u_j \to e_s)),$$

and $Y = (Y_{i,j})_{i \in \{1,...,p\}, j \in \{1,...,r\}}$ is given by

$$Y_{i,j} = P(u_i \to v_j).$$

[2]This is a technical condition on the order in which the start points are connected to the end points in each family of paths.

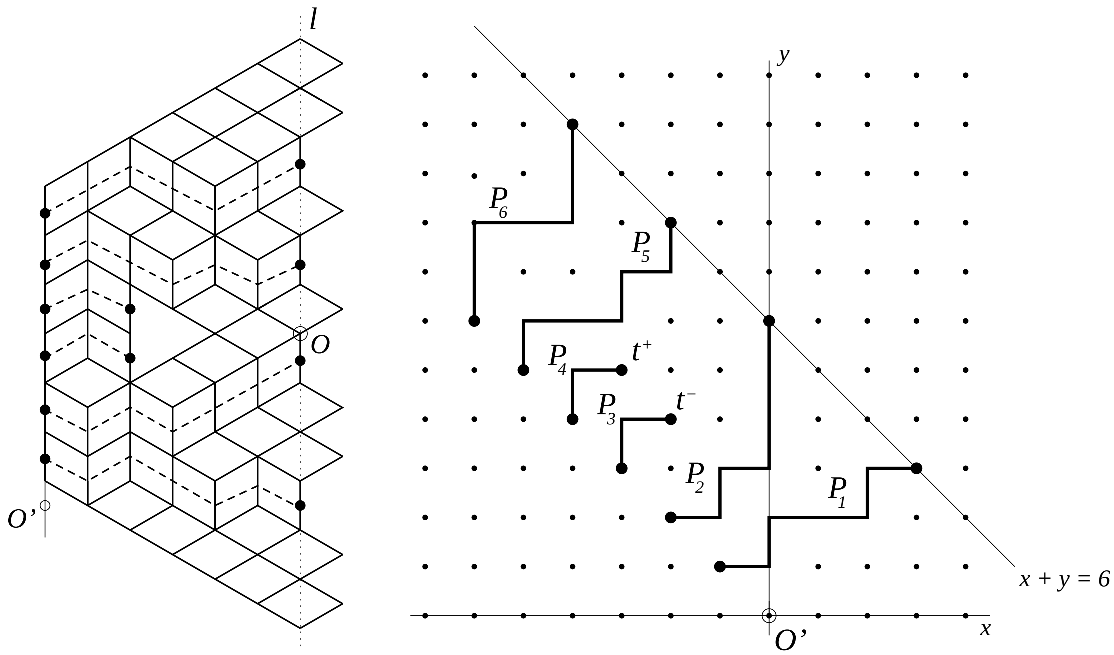

FIGURE 4. Left: the tiling from Figure 2 as lattice paths across dimers. Right: the translation of this tiling to a family of non-intersecting paths in $\mathbb{Z}^2$.

Remark 4.2. In case $p+r$ is odd, a phantom vertex $u_{p+1} = e_{q+1}$ may be adjoined with the property that

$$P(u_s \to e_{q+1}) = \begin{cases} 1, & s = p+1, \\ 0, & otherwise. \end{cases}$$

Then it may be shown that $\mathscr{P}(U \cup u_{p+1} \to (v \oplus E) \cup e_{q+1})$ is given by

$$\mathrm{Pf}\begin{pmatrix} Q & Y \\ -Y^t & 0 \end{pmatrix},$$

where $Y = (Y_{i,j})_{1 \le i \le p+1, 1 \le j \le r}$ is as above with the added condition that $Y_{p+1,j} = 0$ for all j, and $Q = (Q_{i,j})_{1 \le i,j \le p+1}$ is given by

$$Q_{i,j} = \sum_{1 \le s < t \le q+1} (P(u_i \to e_s)P(u_j \to e_t) - P(u_i \to e_t)P(u_j \to e_s)).$$

Remark 4.3. Throughout this article sums in which the starting index is not necessarily smaller than the ending index are interpreted according to the following standard convention:

$$\sum_{r=m}^{n-1} \mathrm{Exp}(r) = \begin{cases} \sum_{r=m}^{n-1} \mathrm{Exp}(r) & n > m \\ 0 & n = m \\ -\sum_{r=n}^{m-1} \mathrm{Exp}(r) & n < m. \end{cases} \tag{4.1}$$

Theorem 4.2 may be used to count families of non-intersecting paths that correspond to vertically symmetric tilings of holey hexagons.

Proposition 4.3. *The number of vertically symmetric tilings of the hexagon* $H_{n,2m}\setminus(\triangleright_k\cup\triangleleft_k)$ *is the Pfaffian of the skew-symmetric matrix* $F=(f_{i,j})_{1\le i,j\le 2m+2}$ *defined by*

$$f_{i,j}=\begin{cases}\sum_{r=i-j+1}^{j-i}\binom{2n}{n+r}, & 1\le i<j\le 2m,\\ \binom{n-k}{(n-k)/2-m+i}, & i\in\{1,\dots,2m\}, j=2m+1,\\ \binom{n-k}{(n-k)/2-m-1+i}, & i\in\{1,\dots,2m\}, j=2m+2,\\ 0, & 2m+1\le i<j\le 2m+2,\end{cases}$$

whilst the number of vertically symmetric tilings of $H_{n,2m-1}\setminus(\triangleright_k\cup\triangleleft_k)$ *is the Pfaffian of the skew-symmetric matrix* $F^*=(f^*_{i,j})_{1\le i,j\le 2m+2}$ *defined by*

$$f^*_{i,j}=\begin{cases}\sum_{r=i-j+1}^{j-i}\binom{2n}{n+r}, & 1\le i<j\le 2m-1,\\ 2^n, & i\in\{1,\dots,2m-1\}, j=2m\\ \binom{n-k}{(n-k+1)/2-m+i}, & i\in\{1,\dots,2m-1\}, j=2m+1,\\ \binom{n-k}{(n-k-1)/2-m+i}, & i\in\{1,\dots,2m-1\}, j=2m+2,\\ 0, & 2m+1\le i<j\le 2m+2.\end{cases}$$

Proof. Consider first the region $H_{n,2m}\setminus(\triangleright_k\cup\triangleleft_k)$. Let $U=A$, $V=\{t^-,t^+\}$ and $E=I\setminus v$, and F be the resulting matrix in Theorem 4.2. Then by the simple fact that $P((a,b)\to(c,d))=\binom{c-a+d-b}{c-a}$, it should be clear that the proposition holds for $j=2m+1$ and $j=2m+2$. For $1\le i<j\le 2m$,

$$\begin{aligned} f_{i,j} &= \sum_{1\le s<t\le 2m+n}\left(\binom{n}{s-i}\binom{n}{t-j}-\binom{n}{s-j}\binom{n}{t-i}\right)\\ &= \sum_{t=1}^{n+2m}\sum_{s=1}^{2m+n}\left(\binom{n}{n-s+i}\binom{n}{s+t-j}-\binom{n}{n-s+j}\binom{n}{s+t-i}\right)\\ &= \sum_{t=1}^{n+2m}\left(\binom{2n}{n+t+i-j}-\binom{2n}{n+t+j-i}\right)\\ &= \sum_{r=i-j+1}^{j-i}\binom{2n}{n+r}, \end{aligned}$$

where the Chu-Vandermonde convolution has been used in the second line.

Next consider the region $H_{n,2m-1}\setminus(\triangleright_k\cup\triangleleft_k)$. Let the point $A_{m+1}=I_{n+m+1}=(-1,0)$ be a phantom vertex as described in Remark 4.2. Let $U=A\cup A_{m+1}$, $V=\{t^-,t^+\}$ and $E=(I\setminus v)\cup I_{n+m+1}$ in Theorem 4.2 and let F^* be the resulting matrix. Then F^* agrees with the matrix F (the determinant of which counts tilings of $V_{n,2m}\setminus\triangleright_k$) everywhere except in the $(2m)$-th row and column, where

$$\begin{aligned} f^*_{i,2m} &= \sum_{s=1}^{n+m}P(u_i\to e_s)\\ &= 2^n, \end{aligned}$$

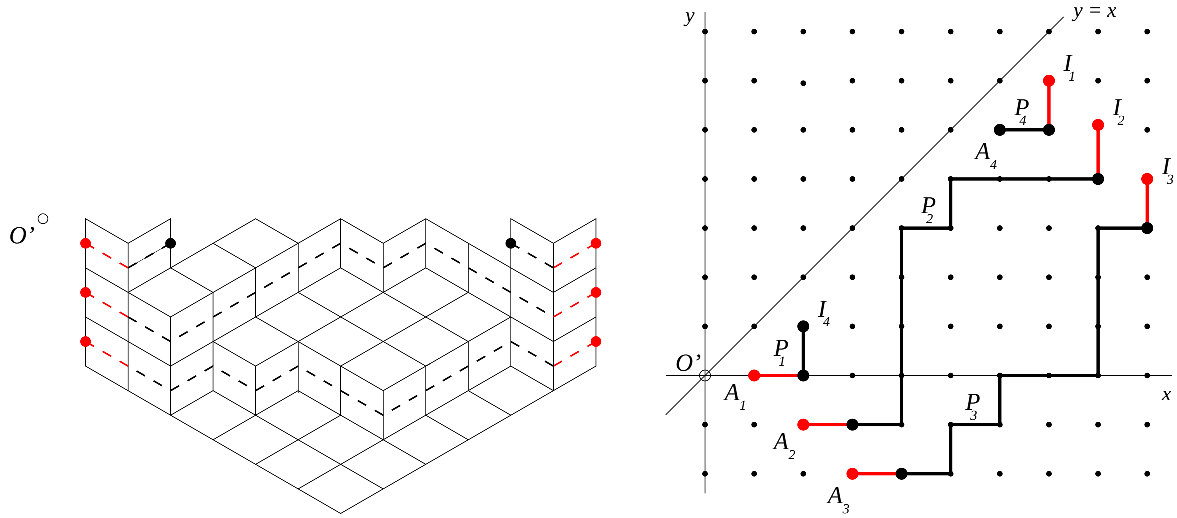

Figure 5. Left: A tiling of $H^-_{6,6} \setminus (\triangleright_4 \cup \triangleleft_4)$. Right: The corresponding set of lattice paths. Lines and dashed lines marked in red highlight forced dimers on the vertical sides.

and

$$f^*_{2m,j} = -\sum_{s=1}^{n+m} P(u_j \to e_s)$$
$$= -2^n,$$

for $1 \leq i, j \leq 2m-1$. This concludes the proof. $\square$

Consider now the region $H^-_{n,2m} \setminus (\triangleright_k \cup \triangleleft_k)$. Observe that in any tiling of $H^-_{n,2m} \setminus (\triangleright_k \cup \triangleleft_k)$ the left and right most vertical sides comprise a set of forced rhombi (Figure 5, left, illustrates such a forcing). Hence it follows that

$$M(H^-_{n-1,2m+1} \setminus (\triangleright_k \cup \triangleleft_k)) = M(H^-_{n,2m} \setminus (\triangleright_k \cup \triangleleft_k)),$$

so it suffices to enumerate tilings of the region $H^-_{n,2m} \setminus (\triangleright_k \cup \triangleleft_k)$.

Consider then the region $H_{n,2m} \setminus (\triangleright_k \cup \triangleleft_k)$ for some integer m. Once again, by translating sets of tilings to sets of lattice paths across dimers, sets of tilings of $H^-_{n,2m} \setminus (\triangleright_k \cup \triangleleft_k)$ correspond to families of non-intersecting paths beginning at a set of start points $A = \bigcup_{s=1}^{m+1} A_s$, where $A_s = (s, 1-s)$ for $s \in \{1, \ldots, m\}$ and $A_{m+1} = ((n+k)/2 + 1, (n+k)/2)$, and ending at $I = \bigcup_{t=1}^{m+1} I_t$, where $I_t = (n+t, n+1-t)$ for $t \in \{1, \ldots, m\}$ and $I_{m+1} = ((n-k)/2+1, (n-k)/2)$, with the condition that no path intersects the line $y = x$ (see Figure 5). With this labelling the number of tilings of $H^-_{n,2m} \setminus (\triangleright_k \cup \triangleleft_k)$ is equal to the number of families of non-intersecting paths $(P_1, P_2, \ldots, P_{m+1})$ where P_i denotes a path from A_i to $I_{\sigma(i)}$ and $\sigma = (1, m+1)$ is the permutation on $m+1$ elements. Such families of paths may be counted directly using the well-known theorem of Lindström-Gessel-Viennot [16], recalled below.

Let $G = (V, E)$ be a locally directed acyclic graph, that is, every vertex has finite degree and G contains no directed cycles. Consider base vertices $B = \{b_1, b_2, \ldots, b_n\}$ and destination vertices $D = \{d_1, d_2, \ldots, d_m\}$ and assign to each directed edge e a weight, ω_e, belonging to some commutative ring. For each directed path $P : b \to d$, let

$\omega(P)$ be the product of the weights of the edges of the path. For any two vertices b and d, let $e(b,d) = \sum_{P:b\to d} \omega(P)$ denote the sum over all paths from b to d. If we assign the weight 1 to each edge then $e(b,d)$ counts the number of paths from b to d. Let $\mathcal{P}$ denote the n-tuple of paths $(P_1, P_2, \ldots, P_n)$ from B to D that satisfy the following properties:

- There exists a permutation σ of $\{1, 2, \ldots, n\}$ such that for every i, P_i is a path running from b_i to $d_{\sigma(i)}$.
- For any $i \neq j$, the paths P_i and P_j have no common vertices.

Given such an n-tuple, denote by $\sigma(\mathcal{P})$ the permutation from the first condition above.

Consider now the matrix N with (i,j)-entry equal to $e(b_i, d_j)$. The Lindström-Gessel-Viennot Theorem then states that the determinant of N is the signed sum over all n-tuples $\mathcal{P} = (P_1, P_2, \ldots, P_n)$ of non-intersecting paths from B to D,

$$\det(N) = \sum_{\mathcal{P}:B\to D} \operatorname{sgn}(\sigma(\mathcal{P})) \prod_{i=1}^{n} \omega(P_i).$$

That is, the determinant of N counts the weights of all sets of non-intersecting paths from B to D, each affected by the sign of the corresponding permutation σ given by P_i taking b_i to $d_{\sigma(i)}$. Indeed if the weights of each edge are 1 and the only permutation possible is the identity then $\det(N)$ counts exactly $\mathscr{P}(B \to D)$ (that is, the number of non-intersecting paths between B and D).

Proposition 4.4. *The number of horizontally symmetric tilings of $H_{n,2m} \setminus (\triangleright_k \cup \triangleleft_k)$ is $(-\det(G))$, where $G = (g_{i,j})_{1\leq i,j\leq m+1}$ is the matrix given by*

$$g_{i,j} = \begin{cases} \binom{2n}{n+j-i} - \binom{2n}{n-j-i+1}, & 1 \leq i,j \leq m, \\ \binom{n-k}{(n-k)/2+1-i} - \binom{n-k}{(n-k)/2-i}, & j = m+1,\ 1 \leq i \leq m, \\ \binom{n-k}{(n-k)/2+1-j} - \binom{n-k}{(n-k)/2-j}, & i = m+1,\ 1 \leq j \leq m, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. In the result of Lindström, Gessel and Viennot replace B with the set A and replace D with the set I (defined previously) and let G be the matrix whose (i,j)-entry is the number of paths from A_i to I_j. The only permutation that gives rise to $(m+1)$-tuples of non-intersecting paths, $\mathcal{P}_{m+1}$, is the permutation $\sigma = (1, m+1)$ on $m+1$ objects. By Lindström-Gessel-Viennot it follows that $\det(G) = -\mathscr{P}(A \to I)$, where $\mathscr{P}(X \to Y)$ is as in Theorem 4.2.

All that remains to be seen is that the entries of G do indeed count the number of paths between A_i and I_j. Care must be taken to ensure that only those paths that do not intersect the line $y = x$ are counted.

Given a starting point a and an ending point b, let $\mathcal{P}$ denote the set of all paths from a to b (including those that cross the line $y = x$). Consider a second set of paths $\mathcal{P}'$ obtained by taking all paths in $\mathcal{P}$ that intersect $y = x$ and reflecting portions between touching points and the last segment of each path in the line $y = x$, thereby obtaining all paths from a to b' (where b' is the reflection of b in $y = x$). Then the difference $|\mathcal{P}| - |\mathcal{P}'|$ gives the total number of paths from a to b that never intersect the line $y = x$.

Suppose $1 \leq i,j \leq m$. Then the number of paths starting at A_i and ending at I_j is given by

$$P(A_i \to I_j) = P((i, 1-i) \to (n+j, n+1-j)).$$

Letting I'_j denote the reflection of I_j in the line $y = x$, the number of paths from A_i to I_j that do not intersect the line $y = x$ is then

$$P(A_i \to I_j) - P(A_i \to I'_j) = \binom{2n}{n+j-i} - \binom{2n}{n-j-i+1}.$$

A similar argument holds for the other entries of G. This concludes proof. $\square$

Precisely the same method may be used to enumerate tilings of the region containing weighted lozenges, $\overline{H}^+_{n,2m} \setminus (\triangleright_k \cup \triangleleft_k)$, with the exception that any path from A to I that intersects the line $y = x$ at t-many points has a total weight of 2^t. It is not hard to convince oneself that $|\mathcal{P}| + |\mathcal{P}'|$ counts all such paths. Such an argument appears in [12]. Proceeding in exactly the same way as Proposition 4.4 gives the following:

Proposition 4.5. *The number of tilings of the weighted region $\overline{H}^+_{n,2m} \setminus (\triangleright_k \cup \triangleleft_k)$ is given by $(-\det(G^+))$, where $G^+ = (g^+_{i,j})_{1\le i,j\le m+1}$ is the matrix defined by*

$$g^+_{i,j} = \begin{cases} \binom{2n}{n-i-j+1} + \binom{2n}{n+i-j}, & 1 \le i,j \le m, \\ \binom{n-k+1}{(n-k)/2+i}, & j = m+1, 1 \le i \le m, \\ \binom{n-k+1}{(n-k)/2+j}, & i = m+1, 1 \le j \le m, \\ 0, & \textit{otherwise}. \end{cases}$$

Remark 4.4. Consider the lattice path representation of $\overline{H}^+_{n,2m} \setminus (\triangleright_k \cup \triangleleft_k)$ defined earlier. A lattice path representation of $\overline{H}^+_{n+1,2m-1} \setminus (\triangleright_k \cup \triangleleft_k)$ may be obtained from that of $\overline{H}^+_{n,2m} \setminus (\triangleright_k \cup \triangleleft_k)$ by extending each start point A_s to the left by 1 and extending each end point I_s vertically by 1 for $s \in \{1, \dots, m\}$. Any tiling of this region will contain forced lozenges along its vertical sides and so

$$M(\overline{H}^+_{n+1,2m-1} \setminus (\triangleright_k \cup \triangleleft_k)) = 2 \cdot M(\overline{H}^+_{n,2m} \setminus (\triangleright_k \cup \triangleleft_k)),$$

where the extra factor of 2 is due to the starting point A_1 lying on the line $y = x$.

Theorems 3.1, 3.2 and 3.3 follow from evaluating the determinants and Pfaffians of the matrices defined above. In the following section these determinants and Pfaffians are evaluated by considering the LU-decomposition of certain matrices.

5. Evaluation of Determinants

This section begins with an extension of Gordon's Lemma [17] that expresses the Pfaffian of the matrix F defined in Theorem 4.3 as the determinant of a much smaller matrix. Similar manipulations reduce the matrix F^* to the same effect. The (unsigned) determinants of these two smaller matrices, as well as those of Proposition 4.4 and Proposition 4.5, are then evaluated by finding their unique LU-decompositions.

Lemma 5.1. *For a positive integer m and a non-negative integer l, let A be the $(2m+2l) \times (2m+2l)$ skew-symmetric matrix of the form*

$$A = \begin{pmatrix} X & Y \\ -Y^t & Z \end{pmatrix},$$

for which the following properties hold:

(i) X is a $2m \times 2m$ matrix such that $X = (x_{j-i})_{1\le i,j\le 2m}$ and $x_{j,i} = -x_{i,j}$;

(ii) $Z = (z_{i,j})_{1\le i,j\le 2l}$ *is a matrix satisfying* $z_{i,j} + z_{i+l,j} + z_{i,j+l} = 0$ *for* $1 \le i, j \le l$, *and* $z_{j,i} = -z_{i,j}$;

(iii) $Y = (y_{i,j})_{1\le i\le 2m, 1\le j\le 2l}$ *is a matrix for which*

$$y_{i,j} = \begin{cases} y_{2m-i,j}, & 1 \le i \le m, 1 \le j \le l, \\ y_{2m+1-i,j-l}, & 1 \le i \le 2m, l+1 \le j \le 2l. \end{cases}$$

Then

$$\mathrm{Pf}(A) = (-1)^{\binom{l}{2}} \det(B),$$

where B *is an* $(m+l) \times (m+l)$ *matrix of the form*

$$B = \begin{pmatrix} \widehat{X} & \widehat{Y}_1 \\ \widehat{Y}_2 & \widehat{Z} \end{pmatrix},$$

the block matrices of which are defined by

$$\begin{aligned}
(\widehat{X})_{i,j} &= x_{i+j-1} + x_{i+j-3} + \cdots + x_{|i-j|+1} && \text{for } 1 \le i, j \le m, \\
(\widehat{Y}_1)_{i,j} &= \sum_{s=0}^{i-1} (y_{m+1-i+2s,j} - y_{m+i-2s,j}) && \text{for } 1 \le i \le m \text{ and } 1 \le j \le l, \\
(\widehat{Y}_2)_{i,j} &= \sum_{s=0}^{j-1} (y_{j+m-2s,i} + y_{m+1-j+2s,i}) && \text{for } 1 \le i \le l \text{ and } 1 \le j \le m, \\
(\widehat{Z})_{i,j} &= z_{i,j+l} + z_{i+l,j+l} && \text{for } 1 \le i, j \le l.
\end{aligned}$$

Proof. Beginning with A, construct a new matrix $A' = (A'_{i,j})_{1\le i,j\le 2m+2l}$ by simultaneously replacing the i-th row of A by the sum

$$\sum_{s=0}^{m-i} (\text{row } i+2s \text{ of A}),$$

and the $(2m+1-i)$-th row of A with

$$\sum_{s=0}^{m-i} (\text{row } 2m+1-i-2s \text{ of A})$$

for $i = 1, ..., m-1$. Perform analogous operations on the columns of the resulting matrix. Note that these operations do not change the value of $\mathrm{Pf}(A)$ and so $\mathrm{Pf}(A) = \mathrm{Pf}(A')$. It follows that for $1 \le i, j \le m$,

$$\begin{aligned}
A'_{i,j} &= \sum_{s=0}^{m-j} \sum_{r=0}^{m-i} x_{j-i+2s-2r} \\
&= \sum_{t=-m}^{m-i-j} (\min\{t+m+1, m-i+1\} - \max\{0, t+j\}) x_{j+i+2t}.
\end{aligned}$$

By assumption X is skew-symmetric, so $x_r = -x_r$ for all $1 \leq r \leq 2m$. One may also verify that

$$\begin{aligned}\min\{t+m+1, m-i+1\} & \\ -\max\{0, t+j\} &= \min\{(-t-i-j)+m+1, m-i+1\} \\ &\qquad -\max\{0, (-t-i-j)+j\},\end{aligned}$$

whence $A'_{i,j}$ vanishes for $1 \leq i, j \leq m$. It is not hard to convince oneself that replacing i and j with $2m-i+1$ and $2m-j+1$ respectively in the above summation gives the (i,j)-entry of A' for $m+1 \leq i, j \leq 2m$, and so these entries also vanish.

For $1 \leq i \leq m$ and $m+1 \leq j \leq 2m$,

$$\begin{aligned}A'_{i,j} &= \sum_{s=0}^{j-m-1} \sum_{r=0}^{m-i} x_{j-i-2r-2s} \\ &= \sum_{t=1}^{j-i} (\min\{t, m-i+1\} - \max\{0, t-j+m\}) x_{i-j+2t}.\end{aligned}$$

It is easy to see that

$$\begin{aligned}\min\{(j-i-t), m-i+1\} & \\ -\max\{0, (j-i-t)-j+m\} &= \min\{t, m-i+1\} \\ &\qquad -\max\{0, t-j+m\}+1,\end{aligned}$$

so again by the skew-symmetry of X,

$$A'_{i,j} = \bar{x}_{i,j},$$

for $1 \leq i \leq m$ and $m+1 \leq j \leq 2m$, where

$$\bar{x}_{i,j} = x_{j-i} + x_{j-i-2} + \cdots + x_{|2m-i-j+1|+1}. \tag{5.1}$$

Having performed exactly the same operations on both the rows and columns of A it follows that A' is also skew-symmetric, thus for $m+1 \leq i \leq 2m$ and $1 \leq j \leq m$, $A'_{i,j} = -A'_{j,i}$.

For the remaining (i,j)-entries of A' it suffices to consider only those entries for which $1 \leq i \leq 2m$ and $j \in \{2m+1, ..., 2m+l\}$. These columns are affected by the row operations alone so for $1 \leq i \leq m$ the (i,j)-entry of A' is

$$\sum_{s=0}^{m-i} y_{i+2s, j-2m},$$

whilst an analogous argument shows that for $m+1 \leq i \leq 2m$, $a'_{i,j}$ is equal to

$$\sum_{s=0}^{i-m-1} y_{i-2s, j-2m}.$$

By the symmetry of Y it should be clear that for $j \in \{2m+l+1, ..., 2m+2l\}$ and $1 \leq i \leq m$,

$$A'_{i,j} = \sum_{s=0}^{m-i} y_{2m+1-i-2s, j-2m-l},$$

whilst for $m+1 \leq i \leq 2m$ and $j \in \{2m+l+1, ..., 2m+2l\}$ the (i,j)-entry of A' is

$$\sum_{s=0}^{i-m-1} y_{2m+1-i+2s,j-2m-l}.$$

Note that A' has now been completely determined since these row and column operations leave Z unchanged.

Construct a new matrix $A'' = (A''_{i,j})_{1\leq i,j\leq 2m+2l}$ by perfoming the following operations on A':

(i) Add column $(2m+l+j)$ to column $(2m+j)$ for $j \in \{1, ..., l\}$ (resp. rows);
(ii) Subtract column $(m+j)$ from column $(m+1-j)$ for $j \in \{1, ..., m\}$ (resp. rows).

Consider the effect of the first set of row and column operations stated above. For $1 \leq i,j \leq 2l$,

$$A''_{i+2m,j+2m} = \begin{cases} 0, & 1 \leq i,j \leq l, \\ z_{i,j} + z_{i+l,j}, & i \in \{1, ..., l\}, j \in \{l+1, ..., 2l\}, \\ z_{i,j} + z_{i,j+l}, & i \in \{l+1, ..., 2l\}, j \in \{1, ..., l\}, \\ z_{i,j}, & l+1 \leq i,j \leq 2l, \end{cases}$$

by the second property in the statement of the lemma.

The (i,j)-entry of A'' for $1 \leq i \leq m$ and $2m+1 \leq j \leq 2m+l$ becomes

$$A''_{i,j} = \sum_{s=0}^{m-i} y_{i+2s,j-2m} + \sum_{s=0}^{m-i} y_{2m+1-i-2s,j-2m},$$

while for $m+1 \leq i \leq 2m$,

$$A''_{i,j} = \sum_{s=0}^{i-m-1} y_{i-2s,j-2m} + \sum_{s=0}^{i-m-1} y_{2m+1-i+2s,j-2m},$$

and crucially (by the third property of the statement of the lemma), $A''_{i,j} = A''_{2m+1-i,j}$ for $1 \leq i \leq m$ and $j \in \{2m+1, ..., 2m+l\}$.

Now consider the effect of the second set of operations applied to A'. Again by the skew-symmetry of A' this leaves all entries unchanged except those for which $i \in \{1, ..., m\}$ and $j \in \{2m+1, ...2m+2l\}$. For $2m+1 \leq j \leq 2m+l$ it should be clear that this entry vanishes, hence the resulting matrix has the form

$$A'' = \begin{pmatrix} 0 & \bar{X} & 0 & Y_1 \\ -\bar{X}^t & 0 & Y_2 & Y_3 \\ 0 & -Y_2^t & 0 & Z_1 \\ -Y_1^t & -Y_3^t & -Z_1^t & Z_2 \end{pmatrix},$$

where for $1 \leq i,j \leq m$,

$$(\bar{X})_{i,j} = \bar{x}_{i,j+m}$$

is given by (5.1) and

$$
\begin{aligned}
(Y_1)_{i,j} &= \sum_{s=0}^{m-i} (y_{2m+1-i-2s,j} - y_{i+2s,j}) && \text{for } i \in \{1, \dots, m\}, j \in \{1, \dots, l\},\\
(Y_2)_{i,j} &= \sum_{s=0}^{i-1} (y_{i+m-2s,j} + y_{m+1-i+2s,j}) && \text{for } i \in \{1, ..., m\}, j \in \{1, ..., l\},\\
(Y_3)_{i,j} &= \sum_{s=0}^{i-1} y_{m+1-i+2s,j} && \text{for } i \in \{1, ..., m\}, j \in \{1, ..., l\},\\
(Z_1)_{i,j} &= z_{i,j+l} + z_{i+l,j+l} && \text{for } 1 \leq i, j \leq l,\\
(Z_2)_{i,j} &= Z_{i+l,j+l} && \text{for } 1 \leq i, j \leq l.
\end{aligned}
$$

By rearranging rows and columns in exactly the same way A'' may be brought into the form

$$
\begin{pmatrix} 0 & 0 & \bar{X} & Y_1 \\ 0 & 0 & -Y_2^t & Z_1 \\ -\bar{X}^t & Y_2 & 0 & Y_3 \\ -Y_1^t & -Z_1^t & -Y_3^t & Z_2 \end{pmatrix}. \tag{5.2}
$$

Since the same operations have been performed on both rows and columns this leaves the Pfaffian of A'' unchanged. By the well-known identity

$$
\mathrm{Pf} \begin{pmatrix} 0 & P \\ -P^t & Q \end{pmatrix} = (-1)^{\binom{n}{2}} \det(P),
$$

where P is an arbitrary $n \times n$ matrix, it follows that

$$
\mathrm{Pf}(A) = \mathrm{Pf}(A'') = (-1)^{\binom{m+l}{2}} \det \begin{pmatrix} \bar{X} & Y_1 \\ -Y_2^t & Z_1 \end{pmatrix}.
$$

Reversing the order of the rows 1 to m and multiplying the last l rows and columns by -1 gives

$$
\begin{aligned}
\mathrm{Pf}(A) &= (-1)^{\left(\binom{m+l}{2} + \binom{m}{2}\right)} \det \begin{pmatrix} \widehat{X} & \widehat{Y}_1 \\ \widehat{Y}_2 & \widehat{Z} \end{pmatrix},\\
&= (-1)^{\binom{l}{2}} \det \begin{pmatrix} \widehat{X} & \widehat{Y}_1 \\ \widehat{Y}_2 & \widehat{Z} \end{pmatrix},
\end{aligned}
$$

where $\widehat{X}, \widehat{Y}_1, \widehat{Y}_2$, and $\widehat{Z}$ are exactly those blocks asserted in the lemma. □

Remark 5.1. Gordon's Lemma may be recovered from the previous result by setting $l = 0$.

Applying Lemma 5.1 directly to the matrix $F = (f_{i,j})_{1 \leq i,j \leq 2m+2}$ from Proposition 4.3 results in the following expression for the signed Pfaffian of F,

$$
\mathrm{Pf}(F) = (-1)^{\binom{l}{2}} \det(\bar{F}), \tag{5.3}
$$

where $\bar{F} = (\bar{f}_{i,j})_{1\le i,j\le m+1}$ is the matrix with (i,j)-entries given by

$$\bar{f}_{i,j} = \begin{cases} f_{i+j-1} + f_{i+j-3} + \cdots + f_{|i-j|+1}, & 1 \le i,j \le m \\ \sum_{s=0}^{i-1}(f_{m+1-i+2s,2m+1} - f_{m+i-2s,2m+1}), & 1 \le i \le m, j = m+1, \\ \sum_{s=0}^{j-1}(f_{j+m-2s,2m+1} + f_{m+1-j+2s,2m+1}), & i = m+1, 1 \le j \le m, \\ 0, & otherwise. \end{cases}$$

Construct one final matrix $\widehat{F} = (\hat{f}_{i,j})_{1\le i,j\le m+1}$ by performing one last set of row and column operations on $\bar{F}$, namely for $i \in \{1,..,m-1\}$, subtract the $(m-i)$-th row from row $(m+1-i)$, and once again perform analogous operations on the columns.

For $1 \le j \le m$,

$$\begin{aligned} \hat{f}_{m+1,j} &= \bar{f}_{m+1,j} - \bar{f}_{m+1,j-1} \\ &= f_{m+j,2m+1} + f_{m+1-j,2m+1} \\ &= \binom{n-k}{(n-k)/2+j} + \binom{n-k}{(n-k)/2+1-j} \\ &= \binom{n-k+1}{(n-k)/2+j}, \end{aligned}$$

which agrees with $g^{+}_{m+1,j}$ (the entries of the matrix whose determinant counts weighted tilings of $\overline{H}^{+}_{n,2m} \setminus (\triangleright_k \cup \triangleleft_k)$) for $1 \le j \le m$.

For $1 \le i,j \le m$,

$$\begin{aligned} \hat{f}_{i,j} &= \bar{f}_{i,j} - \bar{f}_{i-1,j} - (\bar{f}_{i,j-1} - \bar{f}_{i-1,j-1}) \\ &= f_{i+j-1} - f_{i-j} - f_{i+j-2} + f_{i-j+1} \\ &= \binom{2n}{n-i-j+1} + \binom{2n}{n+i-j}, \end{aligned}$$

which again agrees with $g^{+}_{i,j}$ for $1 \le i,j \le m$.

The remaining entries of $\widehat{F}$ are given by

$$\begin{aligned} \hat{f}_{i,m+1} &= \bar{f}_{i,m+1} - \bar{f}_{i-1,m+1} \\ &= f_{m+1-i,2m+1} - f_{m+i,2m+1} \\ &\qquad + 2\sum_{s=1}^{i-1}(f_{m+i+1-2s,2m+1} - f_{m-i+2s,2m+1}) \\ &= \binom{n-k}{(n-k)/2+1-i} - \binom{n-k}{(n-k)/2+i} + 2\sum_{s=2-i}^{i-1}\binom{n-k}{(n-k)/2+s} \\ &= \left(\frac{4(i-1)}{(k-n)} + \frac{2i-1}{i+(n-k)/2}\right)\binom{n-k}{(n-k)/2-i+1}, \end{aligned}$$

where the identity $\sum_{k=0}^{m}\binom{N}{k} = (-1)^m\binom{N-1}{m}$ has been used in the fourth line. Note how none of these operations have changed the value of the determinant, and also that $\hat{f}_{i,j} = g^{+}_{i,j}$ for $1 \le i \le m+1$ and $1 \le j \le m$.

The sign in (5.3) may be ignored since these determinants are considered within the context of counting families of non-intersecting paths, and so

$$M(V_{n,2m} \setminus \triangleright_k) = |\det(\widehat{F})|, \tag{5.4}$$

where $\widehat{F} = (\hat{f}_{i,j})_{1\le i,j\le m+1}$ is the matrix given by

$$\hat{f}_{i,j} = \begin{cases} \binom{2n}{n-i-j+1} + \binom{2n}{n+i-j}, & 1 \le i,j \le m, \\ \left(\frac{4(i-1)}{k-n} + \frac{2i-1}{i+(n-k)/2}\right)\binom{n-k}{(n-k)/2-i+1}, & j = m+1, 1 \le i \le m, \\ \binom{n-k+1}{(n-k)/2+j}, & i = m+1, 1 \le j \le m, \\ 0, & \textit{otherwise.} \end{cases}$$

Theorem 5.2. *The $(m+1)\times(m+1)$ matrix $\widehat{F}$ described above has LU-decomposition*

$$\widehat{F} = \widehat{L} \cdot \widehat{U},$$

where $\widehat{L} = (\hat{l}_{i,j})_{1\le i,j\le m+1}$ has the form

$$\hat{l}_{i,j} = \begin{cases} A_n(i,j), & 1 \le j \le i \le m, \\ B_{n,k}(j), & i = m+1, 1 \le j \le m \\ 1, & i = j = m+1 \\ 0, & \textit{otherwise,} \end{cases}$$

and $\widehat{U} = (\hat{u}_{i,j})_{1\le i,j\le m+1}$ has the form

$$\hat{u}_{i,j} = \begin{cases} C_n(i,j), & 1 \le i \le j \le m, \\ D_{n,k}(i), & j = m+1, 1 \le i \le m, \\ -\sum_{s=1}^{m} B_{n,k}(s)\cdot D_{n,k}(s), & i = j = m+1, \\ 0, & \textit{otherwise,} \end{cases}$$

where

$$\begin{aligned} A_n(i,j) &= \frac{(n)!(i+j-2)!(2j+n-1)!}{(2j-2)!(i-j)!(-i+j+n)!(i+j+n-1)!}, \\ B_{n,k}(j) &= \frac{(-1)^{j+1}(j+n-1)!(2j+n-1)!(n-k+1)!(j+(k+n)/2-2)!}{(j-1)!(2j+2n-1)!((n-k)/2)!((k+n)/2-1)!(j+(n-k)/2)!}, \\ C_n(i,j) &= \frac{(n)!(i+j-2)!(2i+2n-1)!}{(j-i)!(2i+n-2)!(i-j+n)!(i+j+n-1)!}, \\ D_{n,k}(i) &= \frac{(-1)^{i+1}(2i-2)!(i+n-1)!(n-k)!(i+(k+n)/2-2)!}{(i-1)!(2i+n-2)!((n-k)/2)!((k+n)/2-1)!(i+(n-k)/2)!} \\ &\quad + \frac{2(-1)^{i+1}(2i-2)!(i+n)!(n-k)!(i+(k+n)/2-2)!}{(i-2)!(2i+n-2)!((n-k)/2)!((k+n)/2)!(i+(n-k)/2)!}. \end{aligned}$$

Proof. Before embarking on the proof proper observe that the $(m+1,m+1)$-entry of $(\widehat{L}\cdot\widehat{U})$ is 0, whilst for $(i,j) \in \{1,...,m\}^2$,

$$A_n(i,j)\cdot C_n(i,j) = A_n(j,i)\cdot C_n(j,i).$$

The proof then amounts to showing that for $1 \leq i, j \leq m+1$,

$$\sum_{s=1}^{t} \hat{l}_{i,s} \cdot \hat{u}_{s,j} = \hat{f}_{i,j},$$

where $t = \min\{i, j\}$.

This is equivalent proving that the following identities hold:

(i) $\sum_{s=1}^{i} A_n(i,s) \cdot C_n(s,j) = \binom{2n}{n-i-j+1} + \binom{2n}{n+i-j}$ for $1 \leq i \leq j \leq m$.

(ii) $\sum_{s=1}^{i} A_n(i,s) \cdot D_{n,k}(s) = \left(\frac{4(i-1)}{k-n} + \frac{2i-1}{i+(n-k)/2}\right) \binom{n-k}{(n-k)/2-i+1}$ for $1 \leq i \leq m$.

(iii) $\sum_{s=1}^{j} B_{n,k}(s) \cdot C_n(s,j) = \binom{n-k+1}{(n-k)/2+j}$ for $1 \leq j \leq m$.

In order to prove a hypergeometric identity of the form

$$\sum_{k=1}^{n} U(n,k) = S(n,k), \tag{5.5}$$

it suffices to find a suitable recurrence relation that is satisfied by both sides of (5.5). Such a recurrence for the left hand side of (5.5) may easily be found using your favourite implementation of the Zeilberger-Gosper algorithm (see, for example [26]). Then verifying that the right hand side also satisfies such a recurrence and checking initial values reduces to routine computations.

Identity (i) above satisfies the following recurrence

$$\begin{aligned}(-(i-j-n))(i+j-n-1)\sum_{s=1}^{i}(A_n(i,s)\cdot C_n(s,j))\\
+(i^2+i-j^2+j-2n^2-n)\sum_{s=1}^{i+1}(A_n(i+1,s)\cdot C_n(s,j))\\
+(i-j+n+2)(i+j+n+1)\sum_{s=1}^{i+2}(A_n(i+2,s)\cdot C_n(s,j))=0,\end{aligned}$$

whilst the second identity satisfies the recurrence relation

$$\begin{aligned}(i+(k-n)/2-1)(2i^2+2i+(k-n)/2)\sum_{s=1}^{i}A_n(i,s)\cdot D_{n,k}(s)\\
+(2i^2-2i+(k-n)/2)(i+(n-k)/2+1)\sum_{s=1}^{i+1}A_n(i+1,s)\cdot D_{n,k}(s)=0.\end{aligned}$$

The third identity satisfies the recurrence

$$\begin{aligned}(j+(k-n)/2-1)\sum_{s=1}^{j}(B_{n,k}(s)\cdot C_n(s,j))\\
+(j+(n-k)/2+1)\sum_{s=1}^{j+1}(B_{n,k}(s)\cdot C_n(s,j+1))=0,\end{aligned}$$

which completes the proof. □

Proof of Theorem 3.1 part (i). It requires very little work to see that in the previous theorem $A_n(i,i) = 1$ for $i \in \{1, ..., m\}$. Hence the matrix $\widehat{L}$ contains 1s on its diagonal, and so the determinant of $\widehat{F}$ is simply the product of the diagonal entries of $\widehat{U}$, namely

$$\prod_{s=1}^{m} \frac{(n+2s)_n}{(2s-1)_n} \cdot \left(- \sum_{t=1}^{m} B_{n,k}(t) \cdot D_{n,k}(t) \right).$$

The product on the left hand side above is simply a repackaging of $ST(n, 2m)$ and so it follows that

$$M(V_{n,2m} \setminus \triangleright_k) = \left[\sum_{s=1}^{m} B_{n,k}(s) \cdot D_{n,k}(s) \right] \times ST(n, 2m, n).$$

□

Remark 5.2. By letting $k = 0$, the above formula gives the number of vertically symmetric tilings of $H_{n,2m}$ with no rhombi protruding across the vertical line segment of length 2 that intersects the origin of $H_{a,b}$. It appears that these kinds of tilings, where lozenges are forbidden from crossing segments of lattice lines, have not yet been considered in the literature.

Given that $\hat{f}_{i,j} = g^+_{i,j}$ for $i \in \{1, \ldots, m+1\}$ and $j \in \{1, \ldots, m\}$ (and also $\hat{f}_{m+1,m+1} = g^+_{m+1,m+1} = 0$) a little further work yields the following lemma.

Lemma 5.3. *The $(m+1) \times (m+1)$ matrix G^+ has LU-decomposition*

$$G^+ = \widehat{L} \cdot U^+,$$

where $\widehat{L}$ is defined according to Theorem 5.2 and the matrix $U^+ = (u^+_{i,j})_{1 \le i,j \le m+1}$ is given by

$$u^+_{i,j} = \begin{cases} \hat{u}_{i,j}, & 1 \le i \le j \le m, \\ E_{n,k}(i), & 1 \le i \le m, j = m+1, , \\ -\sum_{s=1}^{m} B_{n,k}(s) \cdot E_{n,k}(s), & i = j = m+1. \end{cases}$$

where $E_{n,k}(s)$ is defined to be

$$E_{n,k}(s) = \frac{(-1)^{s+1}(2s-2)!(n-k+1)!(n+s-1)!((k+n)/2+s-2)!}{(s-1)!((n-k)/2)!((k+n)/2-1)!(n+2s-2)!((n-k)/2+s)!}.$$

Proof. It should be immediately clear that the $(m+1, m+1)$-entry of $(\widehat{L} \cdot U^+)$ is 0. Given the aforementioned agreement between $g^+_{i,j}$ and $\hat{f}_{i,j}$ it suffices to show that

$$\sum_{s=1}^{i} A_n(i,s) \cdot E_{n,k}(s) = \binom{n-k+1}{(n-k)/2+i}.$$

A straightforward manipulation gives

$$A_n(i,s) \cdot E_{n,k}(s) = B_{n,k}(s) \cdot C_n(s,i),$$

so summing each side over s from 1 to i gives the result. □

Proof of Theorem 3.3. Since this LU-decomposition is again unique it follows that

$$\det(G^+) = \left[-\sum_{s=1}^{m} B_{n,k}(s) \cdot E_{n,k}(s)\right] \times \prod_{t=1}^{m} \frac{(n+2t)_n}{(2t-1)_n}. \tag{5.6}$$

The product on the right hand side of (5.6) may be re-written so that

$$\begin{aligned} M(\overline{H}^+_{n,2m} \setminus (\triangleright_k \cup \triangleleft_k)) &= \left[\sum_{s=1}^{m} B_{n,k}(s) \cdot E_{n,k}(s)\right] \times ST(n, 2m) \\ &= \frac{1}{2} M(\overline{H}^+_{n+1,2m-1} \setminus (\triangleright_k \cup \triangleleft_k)), \end{aligned}$$

thus concluding the proof. □

Consider now the skew-symmetric matrix $F^* = (f^*_{i,j})_{1\le i,j\le 2m+2}$ defined in Proposition 4.3. Clearly this matrix does not satisfy the conditions of Lemma 5.1 so instead a new matrix $\overline{F^*} = (\bar{f^*}_{i,j})_{1\le i,j\le 2m+2}$ may be constructed by performing the following row and column operations on F^*:

(i) Replace row i with $\sum_{s=0}^{m-i} \text{row}(i+2s)$, and perform analogous operations on the columns.
(ii) Subtract row $(2m+2)$ from row $(2m+1)$, and perform analogous operations on the columns.

In a similar fashion to the proof of Lemma 5.1, for $1 \le i, j \le m$ the (i, j)-entry of $\overline{F^*}$ disappears, whilst for $1 \le i \le m$ and $m+1 \le j \le 2m-1$,

$$\bar{f^*}_{i,j} = \sum_{s=0}^{m-i} f^*_{i+2s,j}.$$

For $1 \le i \le m$, $\bar{f^*}_{i,2m} = (m+1-i)2^n$ and by observing that $f^*_{i,2m+1} = f^*_{2m-i,2m+2}$,

$$\bar{f^*}_{i,2m+1} = \sum_{s=0}^{m-i} \binom{n-k}{(n-k-1)/2 - m + i + 2s} = \bar{f^*}_{i,2m+2}.$$

Hence for $1 \le i \le m$, the entry $\bar{f^*}_{i,2m+1}$ vanishes once the second set of row and column operations have been applied, whilst for $m+1 \le i \le 2m-1$,

$$\begin{aligned} \bar{f^*}_{i,2m+1} &= \binom{n-k}{(n-k+1)/2 - m + i} - \binom{n-k}{(n-k-1)/2 - m + i} \\ &= \frac{2(m-i)}{n-k+1} \binom{n-k+1}{(n-k+1)/2 - m + i}. \end{aligned}$$

Since F^* is skew-symmetric and exactly the same row and column operations have been applied to F^* in order to construct $\overline{F^*}$, the matrix $\overline{F^*}$ has now been determined completely. By reversing the order of the first m rows and columns, and then interchanging rows and columns in the correct way, $\overline{F^*}$ may be brought into the form

$$\begin{pmatrix} 0 & P \\ -P^T & * \end{pmatrix},$$

where $P = (P_{i,j})_{1\le i,j\le m+1}$ is the matrix with entries given by

$$P_{i,j} = \begin{cases} \sum_{s=0}^{i-1} f^*_{m+1-i+2s,j+m}, & ,i \in \{1,\dots,m\}, j \in \{1,...,m-1\}, \\ 2^n \cdot i, & j = m, i \in \{1,...,m\}, \\ \sum_{s=0}^{i-1} \binom{n-k}{(n-k+1)/2-i+2s}, & j = m+1, 1 \le i \le m, \\ \frac{2j}{n-k+1}\binom{n-k+1}{(n-k+1)/2-j}, & i = m+1, j \in \{1,\dots,m\}. \end{cases}$$

Then clearly $|\operatorname{Pf}(\overline{F^*})| = |\det(P)|$. Proceed by constructing a final matrix $\widehat{F^*} = (\hat{f}^*_{i,j})_{1\le i,j\le m+1}$ from P by subtracting row $(m-i)$ from row $(m+1-i)$ for $i \in \{1,\dots,m-1\}$, and similarly for $j \in \{2,\dots,m-1\}$ subtract column $(m-j)$ from column $(m+1-j)$.

For $i \in \{1,\dots,m\}$, $\hat{f}^*_{i,m} = 2^n(i+1-i) = 2^n$, while

$$\begin{aligned} \hat{f}^*_{i,m+1} &= \sum_{s=0}^{i-1}\binom{n-k}{(n-k+1)/2-i+2s} - \sum_{s=0}^{i-2}\binom{n-k}{(n-k+3)/2-i+2s} \\ &= \sum_{s=0}^{2i-2}(-1)^s\binom{n-k}{(n-k+1)/2-i+s} \\ &= \binom{n-k}{(n-k+1)/2-i}, \end{aligned}$$

where the identity $\sum_{k=0}^{m}(-1)^k\binom{A}{k} = (-1)^m\binom{A-1}{m}$ has been used in the second line. For $1 \le j \le m-1$,

$$\hat{f}^*_{m+1,j} = \frac{2j}{n-k+1}\binom{n-k+1}{(n-k+1)/2-j} - \frac{2(j-1)}{n-k+1}\binom{n-k+1}{(n-k+1)/2-j+1},$$

and also $\hat{f}^*_{m+1,m} = P_{m+1,m}$.

Lastly for $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,m-1\}$,

$$\begin{aligned} \hat{f}^*_{i,j} &= (P_{i,j} - P_{i,j-1}) - (P_{i-1,j} - P_{i-1,j-1}) \\ &= \sum_{s=0}^{i-1}(f^*_{m+1-i+2s,j+m} - f^*_{m+1-i+2s,j+m-1}) \\ &\qquad - \sum_{s=0}^{i-2}(f^*_{m+2-i+2s,j+m} - f^*_{m+2-i+2s,j+m-1}) \\ &= \binom{2n}{n+i-j} + \binom{2n}{n-i-j+1} \\ &= \hat{f}_{i,j}. \end{aligned}$$

Thus it follows that

$$M(V_{n,2m-1} \setminus \triangleright_k) = |\det(\widehat{F^*})|,$$

where $\widehat{F^*} = (\hat{f}^*_{i,j})_{1\le i,j\le m+1}$ is the matrix given by by

$$\hat{f}^*_{i,j} = \begin{cases} \hat{f}_{i,j}, & i\in\{1,\dots,m\}, j\in\{1,\dots,m-1\},\\ 2^n, & j=m, i\in\{1,\dots,m\},\\ \binom{n-k}{(n-k+1)/2-i}, & j=m+1, 1\le i\le m,\\ \frac{2j}{n-k+1}\binom{n-k+1}{(n-k+1)/2-j} - \frac{2(j-1)}{n-k+1}\binom{n-k+1}{(n-k+3)/2-j}, & i=m+1, j\in\{1,\dots,m-1\},\\ \frac{2m}{n-k+1}\binom{n-k+1}{(n-k+1)/2-m}, & i=m+1, j=m\\ 0, & \textit{otherwise.}\end{cases}$$

Theorem 5.4. *The matrix $\widehat{F^*}$ defined above has LU-decomposition*

$$\widehat{F^*} = L^*\cdot U^*$$

where $L^* = (l^*_{i,j})_{1\le i,j\le m+1}$ is defined as

$$l^*_{i,j} = \begin{cases} A_n(i,j), & 1\le j\le i\le m,\\ B^*_{n,k}(j), & i=m+1, 1\le j\le m\\ \frac{1}{D^*_n(m)}\left(-\sum_{s=1}^{m-1} D^*_n(s)\cdot B^*_{n,k}(s)\right), & \imath = m+1, j=m\\ 1 & i=j=m+1\\ 0, & \textit{otherwise.}\end{cases}$$

and $U^* = (u^*_{i,j})_{1\le i,j\le m+1}$ is given by

$$u^*_{i,j} = \begin{cases} C_n(i,j), & 1\le i\le j\le m-1,\\ D^*_n(i), & j=m, 1\le i\le m,\\ E^*_{n,k}(i), & j=m+1, 1\le i\le m,\\ P^*_{n,k}(m), & i=j=m+1,\\ 0, & \textit{otherwise,}\end{cases}$$

with $A_n(i,j)$ and $C_n(i,j)$ as before and

$$B^*_{n,k}(j) = \frac{2(-1)^{j+1}(j+n-1)!(2j+n-1)!(n-k)!\left(\frac{1}{2}(2j+k+n-5)\right)!}{(j-1)!(2j+2n-1)!\left(\frac{1}{2}(-k+n-1)\right)!\left(\frac{1}{2}(k+n-3)\right)!\left(\frac{1}{2}(2j-k+n+1)\right)!}$$
$$+\frac{4(-1)^{j+1}(j+n)!(2j+n-1)!(n-k)!\left(\frac{1}{2}(2j+k+n-5)\right)!}{(j-2)!(2j+2n-1)!\left(\frac{1}{2}(-k+n-1)\right)!\left(\frac{1}{2}(k+n-1)\right)!\left(\frac{1}{2}(2j-k+n+1)\right)!}$$

$$D^*_n(i) = \frac{2^n(2i-2)!(i+n-1)!}{(i-1)!(2i+n-2)!},$$

$$E^*_{n,k}(i) = \frac{(-1)^{i+1}(2i-2)!(i+n-1)!(n-k)!\left(\frac{1}{2}(2i+k+n-3)\right)!}{(i-1)!(2i+n-2)!\left(\frac{1}{2}(-k+n-1)\right)!\left(\frac{1}{2}(k+n-1)\right)!\left(\frac{1}{2}(2i-k+n-1)\right)!}$$

$$P^*_{n,k}(m) = \sum_{s=1}^{m-1}\left(\frac{D^*_n(s)E^*_{n,k}(m)}{D_n(m)} - E^*_{n,k}(s)\right)\cdot B^*_{n,k}(s).$$

Proof. It should be clear that the (i,j)-entry of $L^*\cdot U^*$ is $\hat{f}^*$ for $(i,j)\in\{(m+1,m),(m+1,m+1)\}$ and since $\hat{f}^*_{i,j} = \hat{f}_{i,j}$ for $(i,j)\in\{1,\dots,m\}\times\{1,\dots,m-1\}$, proof of this theorem reduces to showing that the following identities hold:

(i) $\sum_{s=1}^{i} A_n(i,s)\cdot D_n^*(s) = 2^n$ for $i\in\{1,\dots,m\}$;
(ii) $\sum_{s=1}^{i} A_n(i,s)\cdot E_{n,k}^*(s) = \binom{n-k}{(n-k+1)/2-i}$ for $i\in\{1,\dots,m\}$;
(iii) $\sum_{s=1}^{j} B_{n,k}^*(s)\cdot C_n(s,j) = \frac{2j}{n-k+1}\binom{n-k+1}{(n-k+1)/2-j} - \frac{2(j-1)}{n-k+1}\binom{n-k+1}{(n-k+3)/2-j}$ for $j\in\{1,\dots m-1\}$.

Identity (i) may be written as a hypergeometric series (an explanation as to how this may be done is postponed until Section 6)

$$\sum_{s=1}^{i} A_n(i,s)\cdot D_n^*(s) = {}_4F_3\left[\begin{matrix} n+1, & (n+3)/2, & 1-i, & i \\ (n+1)/2, & n+1+i, & n-i+2 \end{matrix}; -1\right]\times \frac{2^n(n+1)(n!)^2}{(n-i+1)!(n+i)!}. \quad (5.7)$$

By applying the following summation formula to (5.7) (see [29, (2.3.4.6); Appendix (III.10)]),

$${}_4F_3\left[\begin{matrix} a, & a/2+1, & b, & c \\ a/2, & a-b+1, & a-c+1 \end{matrix}; -1\right] = \frac{\Gamma(a-b+1)\Gamma(a-c+1)}{\Gamma(a+1)\Gamma(a-b-c+1)},$$

the right hand side of (5.7) may be transformed into

$$\frac{2^n(n+1)(n!)^2}{(n-i+1)!(n+i)!}\cdot\frac{\Gamma(n-i+2)\Gamma(i+n+1)}{\Gamma(n+1)\Gamma(n+2)} = 2^n.$$

Proof of the remaining identities follows in much the same way as Theorem 5.2. The left hand side of identity (ii) satisfies the following recurrence

$$(2i+k-n-1)\sum_{s=1}^{i} A_n(i,s)\cdot E_{n,k}^*(s) + (2i-k+n+1)\sum_{s=1}^{i+1} A_n(i+1,s)\cdot E_{n,k}^*(s) = 0,$$

whilst the left hand side of identity (iii) satisfies

$$(2j+k-n-3)(4j^2+4j+k-n-1)\sum_{s=1}^{j} B_{n,k}^*(s)\cdot C_n(s,j) + (4j^2-4j+k-n-1)(2j-k+n+3)\sum_{s=1}^{j+1} B_{n,k}^*(s)\cdot C_n(s,j+1) = 0.$$

After verifying the right hand side of these identities also satisfy the respective recurrence relations and checking initial conditions the proof is complete. □

Proof of Theorem 3.1 part (ii). Again it is easy to see that $l_{i,i}^* = 1$ for $i\in\{1,\dots,m+1\}$ so it follows that

$$\det(\widehat{F^*}) = \left(\prod_{s=1}^{m} C_n(s,s)\right)\cdot D_n^*(m)\cdot P_{n,k}^*(m).$$

The product above may be re-written as

$$M(V_{n,2m-1}\setminus \triangleright_k) = P_{n,k}^*(m)\times ST(n,2m-1),$$

thus completing the proof of Theorem 3.1. □

Remark 5.3. Note that the product in Theorem 3.1 is a re-packaging of the formula for the number of symmetric plane partitions in a $n \times 2m$ box. The appearance of MacMahon's original formula for vertically symmetric tilings in Theorem 3.1 is not surpising if one considers the translation of such plane partitions to non-intersecting lattice paths. After a little thought one sees that these may be counted by the determinant of the $(m \times m)$ submatrix obtained by taking rows (resp. columns) $1, \dots, m$ of the matrix $\widehat{F}$ (or indeed $\widehat{F^*}$) defined above. The determinant of this submatrix is simply $\prod_{i=1}^{m} \hat{u}_{i,i} = \prod_{i=1}^{m} u^*_{i,j} = ST(n, 2m)$.

What remains is to find an explicit formula for the determinant of the matrix G from Proposition 4.4. The LU-decomposition of this matrix follows below.

Theorem 5.5. *The matrix G whose determinant counts the number of horizontally symmetric tilings of $H_{n,2m} \setminus (\triangleright_k \cup \triangleleft_k)$ has LU-decomposition*

$$G = L \cdot U,$$

where $L = (l_{i,j})_{1 \le i,j \le m+1}$ is given by

$$l_{i,j} = \begin{cases} A'_n(i,j), & 1 \le j \le i \le m, \\ B'_{n,k}(j), & i = m+1, 1 \le j \le m, \\ 1, & i = j = m+1 \\ 0, & \text{otherwise}, \end{cases}$$

and $U = (u_{i,j})_{1 \le i,j \le m+1}$ is given by

$$u_{i,j} = \begin{cases} C'_n(i,j), & 1 \le i \le j \le m, \\ D'_{n,k}(i), & j = m+1, 1 \le i \le m, \\ -\sum_{s=1}^{m} B'_{n,k}(s) \cdot D'_{n,k}(s), & i = j = m+1, \\ 0, & \text{otherwise}. \end{cases}$$

where

$$\begin{aligned} A'_n(i,j) &= \frac{n!(i+j-2)!(2j+n-1)!(2i-1)}{(2j-1)!(i-j)!(j-i+n)!(i+j+n-1)!}, \\ B'_{n,k}(j) &= \frac{(-1)^{j-1}(j+n-2)!(2j+n-1)!(n-k)!\left(j+\frac{k}{2}+\frac{n}{2}-2\right)!}{2(j-1)!(2j+2n-3)!\left(\frac{n}{2}-\frac{k}{2}\right)!\left(\frac{k}{2}+\frac{n}{2}-1\right)!\left(j-\frac{k}{2}+\frac{n}{2}\right)!}, \\ C'_n(i,j) &= \frac{(2j-1)n!(i+j-2)!(2i+2n-2)!}{(j-i)!(2i+n-2)!(i-j+n)!(i+j+n-1)!}, \\ D'_{n,k}(i) &= \frac{(-1)^{i+1}(2i)!(i+n-1)!(n-k)!\left(i+\frac{k}{2}+\frac{n}{2}-2\right)!}{2(i!)(2i+n-2)!\left(\frac{k}{2}+\frac{n}{2}-1\right)!\frac{n-k}{2}!\left(i-\frac{k}{2}+\frac{n}{2}\right)!}. \end{aligned}$$

Proof. It should be immediately obvious that the $(m+1, m+1)$-entry of $(L \cdot U)$ is 0, after a little further thought one sees that

$$A'_n(i,j) \cdot C'_n(i,j) = A'_n(j,i) \cdot C'_n(j,i).$$

Straightforward manipulations give

$$A'_n(p,q) \cdot D'_{n,k}(q) = C'_n(p,q) \cdot B'_{n,k}(q),$$

hence it suffices to consider only those entries of $L_g \cdot U_g$ for which $1 \leq i \leq j \leq m+1$.

The approach here is the same as in the proof of Theorem 5.2, that is, a set of recurrence relations are given that satisfy each side of the following identities:

(i) $\sum_{s=1}^{i} A'_n(i,s) \cdot C'_n(s,j) = \binom{2n}{n+j-i} - \binom{2n}{n-j-i+1}$;

(ii) $\sum_{s=1}^{i} A'_n(i,s) \cdot D'_{n,k}(s) = \binom{n-k}{(n-k)/2+1-i} - \binom{n-k}{(n-k)/2-i}$.

Again by using your favourite implementation of the Zeilberger-Gosper algorithm it is possible to show that the left hand side of the first identity above satisfies the following recurrence

$$\begin{aligned}(j+n-i)(i+j-n-1)\sum_{s=1}^{i} A'_n(i,s)\cdot C'_n(s,j)\\ -2(i^2+i-j^2-j-n^2-n)\sum_{s=1}^{i+1} A'_n(i+1,s)\cdot C'_n(s,j)\\ -(i+j+n+1)(i-j+n+2)\sum_{s=1}^{i+2} A'_n(i+2,s)\cdot C'_n(s,j)=0,\end{aligned}$$

whilst the second identity satisfies

$$\begin{aligned}(2i+1)(2i+k-n-2)\sum_{s=1}^{i} A'_n(i,s)\cdot D'_{n,k}(s)\\ +(2i-1)(2i+n-k+2)\sum_{s=1}^{i+1} A'_n(i+1,s)\cdot D'_{n,k}(s)=0.\end{aligned}$$

Verifying that the right hand side of (i) and (ii) satisfy the corresponding recurrence relations and checking intial conditions is simply a routine computation. This completes the proof. □

Proof of Theroem 3.2. Once again the LU-decomposition above is unique, it therefore follows that

$$\det(G) = \left[-\sum_{s=1}^{m} B'_{n,k}(s)\cdot D'_{n,k}(s)\right] \times \prod_{t=1}^{m} \frac{(n+2t-1)_n}{(2t)_n}.$$

The product on the right hand side may be re-written as Proctor's formula [30], giving

$$\begin{aligned}M(H^-_{n,2m}\setminus(\triangleright_k\cup\triangleleft_k)) &= \left[\sum_{s=1}^{m} B'_{n,k}(s)\cdot D'_{n,k}(s)\right]\times TC(n,2m)\\ &= M(H^-_{n-1,2m+1}\setminus(\triangleright_k\cup\triangleleft_k)).\end{aligned}$$

□

Remark 5.4. Combining Theorem 3.2 with Theorem 3.3 by way of Proposition 2.4 yields Theorem 3.4.

This concludes the proofs of Theorem 3.1, Theorem 3.2 and Theorem 3.4. The asymptotics of the correlation functions defined in Section 2 may now be derived using these exact enumeration formulas.

6. Asymptotic Analysis

This final section is devoted to the proofs of Theorems 2.1, 2.2, and 2.3. The exact enumeration formulas from Section 3 are inserted into the corresponding correlation functions defined in Section 2 and asymptotic expressions for the different interactions are derived. For the sake of simplicity the formulas that enumerate tilings of $V_{a,b} \setminus \triangleright_k$, $\overline{H}^+_{a,b} \setminus (\triangleright_k \cup \triangleleft_k)$ and $H^-_{a,b} \setminus (\triangleright_k \cup \triangleleft_k)$ for even b are used in the corresponding correlation functions, though of course the results obtained here agree with the case for odd b.

Recall the correlation function $\omega_V(k;\xi)$ defined in Section 2,

$$\omega(k;\xi) = \lim_{n\to\infty} \frac{M(V_{n,2m} \setminus \triangleright_k)}{M(V_{n,2m})},$$

where $m \sim \xi n/2$ and $M(R)$ denotes the number of tilings of the region R. The enumeration formula for $M(V_{n,2m} \setminus \triangleright_k)$ given in Section 3 lends itself readily to asymptotic analysis of $\omega_V(k;\xi)$ since it is of the form $M(V_{n,2m} \setminus \triangleright_k) = M(V_{n,2m}) \cdot \sum_{s=1}^{m} B_{n,k}(s) \cdot D_{n,k}(s)$. It follows that

$$\omega_V(k;\xi) = \lim_{n\to\infty} \left(\sum_{s=1}^{m} B_{n,k}(s) \cdot D_{n,k}(s) \right).$$

Similar arguments hold for $\omega_{H^-}(k;\xi)$ and $\omega_{H^+}(k;\xi)$. The product of these two functions then gives $\omega_H(k;\xi)$ according to the Matchings Factorization Theorem [7].

In order to extract asymptotic expressions for these correlation functions the finite sums from Theorems 3.1, 3.2 and 3.3 are first re-written as limits of hypergeometric series. By applying some transformation formulas and sending first n and m to infinity, one obtains these correlation functions in terms of k (which parametrises the distance between the holes). Asymptotic expressions for the different interactions are then derived by letting k become large. Before embarking on the proofs proper, some standard definitions and necessary transformation formulas concerning hypergeometric series are recalled below.

Consider the ${}_pF_q$ hypergeometric series

$${}_pF_q\left[\begin{matrix} a_1, & a_2, & \dots, & a_p \\ b_1, & b_2, & \dots, & b_q \end{matrix}; z\right] = \sum_{s=0}^{\infty} \frac{(a_1)_s (a_2)_s \cdots (a_p)_s}{(b_1)_s (b_2)_s \cdots (b_q)_s} \frac{z^s}{s!},$$

where $(\alpha)_\beta$ denotes the Pochhammer symbol, that is,

$$(\alpha)_\beta = \begin{cases} (\alpha)_\beta = (\alpha)(\alpha+1)\cdots(\alpha+\beta-1), & \beta \neq 0, \\ 1, & \beta = 0. \end{cases}$$

A hypergeometric series is considered *well-poised* if $a_1 + 1 = b_1 + a_2 = \cdots = b_q + a_p$. A *very well-poised* hypergeometric series is a well-poised hypergeometric series for which $a_2 = a_1/2 + 1$.

It is often convenient to re-write a finite sum with summands that consist of hypergeometric terms as a limit hypergeometric series by introducing a term that "artificially" terminates the series after a particular point. Consider, for example, the following finite sum,

$$\sum_{s=0}^{t-1} = \frac{(a_1)_s (a_2)_s \cdots (a_p)_s}{(b_1)_s (b_2)_s \cdots (b_q)_s} \frac{z^s}{s!}. \tag{6.1}$$

By multiplying the summand by $(1-t)_s/(1-t+\epsilon)_s$ it is possible to re-write this finite sum as a limit of a hypergeometric series:

$$\sum_{s=0}^{t-1}\frac{(a_1)_s(a_2)_s\cdots(a_p)_s}{(b_1)_s(b_2)_s\cdots(b_q)_s}\frac{z^s}{s!}=\lim_{\epsilon\to 0}\left(\sum_{s=0}^{\infty}\frac{(a_1)_s(a_2)_s\cdots(a_p)_s(1-t)_s}{(b_1)_s(b_2)_s\cdots(b_q)_s(1-t+\epsilon)_s}\frac{z^s}{s!}\right)$$
$$=\lim_{\epsilon\to 0}\left({}_{p+1}F_{q+1}\left[\begin{matrix}a_1,\ a_2,\ \ldots,\ a_p,\ 1-t+\epsilon,\\ b_1,\ b_2,\ \ldots,\ d_q,\ 1-t\end{matrix};z\right]\right).$$

Suppose in addition that the terms $a_1,\ldots,a_p$ and $b_1,\ldots,b_q$ satisfy the conditions for a very well-poised hypergeometric series. Then by multiplying the summand in (6.1) by a factor of

$$\frac{(1-t)_s(a_1+t)_s}{(a_1+t+\epsilon)_s(1-t+\epsilon)_s},$$

the finite sum (6.1) may be expressed as a limit of a very well-poised ${}_{p+2}F_{q+2}$ hypergeometric series

$$\lim_{\epsilon\to 0}\left(\sum_{s=0}^{\infty}\frac{(a_1+\epsilon)_s(a_2+\epsilon/2)_s\cdots(a_p+\epsilon/2)_s(1-t)_s(a_1+t)_s}{(b_1+\epsilon/2)_s(b_2+\epsilon/2)_s\cdots(b_q+\epsilon/2)_s(a_1+t+\epsilon)_s(1-t+\epsilon)_s}\frac{z^s}{s!}\right)$$
$$=\lim_{\epsilon\to 0}\left({}_{p+2}F_{q+2}\left[\begin{matrix}a_1+\epsilon,\ \ a_2+\epsilon/2,\ \ \ldots,\ \ a_p+\epsilon/2,\ \ 1-t,\ \ a_1+t\\ b_1+\epsilon/2,\ \ b_2+\epsilon/2,\ \ \ldots,\ \ b_q+\epsilon/2,\ \ a_1+t+\epsilon,\ \ 1-t+\epsilon\end{matrix};z\right]\right).$$

There are a wealth of transformation and summation formulas that may be applied to hypergeometric series. Throughout this section frequent use shall be made of the following transformation formula for a ${}_7F_6$ series (see [29, (2.4.1.1), reversed])

$${}_7F_6\left[\begin{matrix}a,\ a/2+1,\ b,\ c,\ d,\ e,\ -n\\ a/2,\ a-b+1,\ a-c+1,\ a-d+1,\ a-e+1,\ a+n+1\end{matrix};1\right]=$$
$$\frac{(a+1)_n(a-d-e+1)_n}{(a-d+1)_n(a-e+1)_n}{}_4F_3\left[\begin{matrix}a-b-c+1,\ d,\ e,\ -n\\ a-b+1,\ a-c+1,\ -a+d+e-n\end{matrix};1\right]. \tag{6.2}$$

Another transformation formula which will be useful is the following (see [29, (1.8.10)])

$${}_2F_1\left[\begin{matrix}-n,\ a\\ c\end{matrix};z\right]=\frac{(1-z)^n(a)_n}{(c)_n}{}_2F_1\left[\begin{matrix}-n,\ c-a\\ -a-n+1\end{matrix};(1-z)^{-1}\right]. \tag{6.3}$$

Proof of Theorem 2.2. Consider the region $V_{n,2m}\setminus\triangleright_k$. According to Section 2 the correlation function of this triangular hole with the free boundary to the right of $V_{n,2m}\setminus\triangleright_k$ is

$$\omega_V(k;\xi)=\lim_{n\to\infty}\left(\sum_{s=1}^{m}B_{n,k}(s)\cdot D_{n,k}(s)\right), \tag{6.4}$$

where $m \sim \xi n/2$ and

$$B_{n,k}(s) = \frac{(-1)^{s+1}(-k+n+1)!(n+s-1)!(n+2s-1)!\left(\frac{k}{2}+\frac{n}{2}+s-2\right)!}{(s-1)!\left(\frac{n}{2}-\frac{k}{2}\right)!\left(\frac{k}{2}+\frac{n}{2}-1\right)!(2n+2s-1)!\left(-\frac{k}{2}+\frac{n}{2}+s\right)!},$$

$$D_{n,k}(s) = \frac{(-1)^{s+1}(2s-2)!(n-k)!(n+s-1)!\left(\frac{k}{2}+\frac{n}{2}+s-2\right)!}{(s-1)!\left(\frac{n}{2}-\frac{k}{2}\right)!\left(\frac{k}{2}+\frac{n}{2}-1\right)!(n+2s-2)!\left(-\frac{k}{2}+\frac{n}{2}+s\right)!} + \frac{2(-1)^{s+1}(2s-2)!(n-k)!(n+s)!\left(\frac{k}{2}+\frac{n}{2}+s-2\right)!}{(s-2)!\left(\frac{n}{2}-\frac{k}{2}\right)!\left(\frac{k}{2}+\frac{n}{2}\right)!(n+2s-2)!\left(-\frac{k}{2}+\frac{n}{2}+s\right)!}.$$

The expression on the right of (6.4) may be written as two separate finite sums consisting of purely hypergeometric terms. These finite sums may in turn be expressed as a limit of two very well-poised hypergeometric series that have been artificially terminated and so (6.4) may be re-written in the following way:

$$\begin{aligned}\omega_V(k;\xi) = \lim_{n\to\infty}\lim_{\epsilon\to 0}\Bigg(&\frac{n!(n+1)!(n-k)!(n-k+1)!}{(2n+1)!((n-k)/2)!^2((n-k)/2+1)!^2}\\ &\times {}_7F_6\left[\begin{matrix} n+1+\epsilon, & \frac{n+3+\epsilon}{2}, & \frac{k+n+\epsilon}{2}, & \frac{k+n+\epsilon}{2}, & \frac{1+\epsilon}{2}, & n+m+1, & 1-m \\ \frac{n+1+\epsilon}{2}, & \frac{n-k+4+\epsilon}{2}, & \frac{n-k+4+\epsilon}{2}, & \frac{2n+3+\epsilon}{2}, & 1-m+\epsilon, & n+m+1+\epsilon \end{matrix};1\right]\\ &+\frac{4(n+1)!(n+3)!((n+k)/2)!(n-k)!(n-k+1)!}{(2n+3)!((n-k)/2)!^2((n-k)/2+2)!^2((n+k)/2-1)!}\\ &\times {}_7F_6\left[\begin{matrix} n+3+\epsilon, & \frac{n+5+\epsilon}{2}, & \frac{k+n+2+\epsilon}{2}, & \frac{k+n+2+\epsilon}{2}, & \frac{3+\epsilon}{2}, & n+m+2, & 2-m \\ \frac{n+3+\epsilon}{2}, & \frac{n-k+6+\epsilon}{2}, & \frac{n-k+6+\epsilon}{2}, & \frac{2n+5+\epsilon}{2}, & 2-m+\epsilon, & n+m+2+\epsilon \end{matrix};1\right]\Bigg). \end{aligned} \tag{6.5}$$

Applying (6.2) separately to each ${}_7F_6$ series in the above sum gives

$$\frac{(n+2+\epsilon)_{m-1}((1+\epsilon)/2-m)_{m-1}}{(n+(3+\epsilon)/2)_{m-1}(1-m+\epsilon)_{m-1}}\,{}_4F_3\left[\begin{matrix} 2-k, & \frac{1+\epsilon}{2}, & m+n+1, & 1-m \\ & \frac{n-k+4+\epsilon}{2}, & \frac{n-k+4+\epsilon}{2}, & \frac{3-\epsilon}{2} \end{matrix};1\right] \tag{6.6}$$

for the upper hypergeometric series in (6.5) and

$$\frac{(n+4+\epsilon)_{m-2}((1+\epsilon)/2-m)_{m-2}}{(n+(5+\epsilon)/2)_{m-2}(2-m+\epsilon)_{m-2}}\,{}_4F_3\left[\begin{matrix} 2-k, & \frac{3+\epsilon}{2}, & m+n+2, & 2-m \\ & \frac{n-k+6+\epsilon}{2}, & \frac{n-k+6+\epsilon}{2}, & \frac{5-\epsilon}{2} \end{matrix};1\right] \tag{6.7}$$

for the lower. Combining (6.6) with the corresponding pre-factor in (6.5) and letting ϵ tend to zero gives gives

$$\frac{(n-k)!(n-k+1)!(n+m)!(2m-1)!(n+m-1)!}{((n-k)/2)!^2((n-k)/2+1)!^2(m-1)!^2(2n+2m-1)!}\times \\ {}_4F_3\left[\begin{matrix} 2-k, & 1/2, & m+n+1, & 1-m \\ (n-k+4)/2, & (n-k+4)/2, & 3/2 \end{matrix};1\right]. \tag{6.8}$$

Applying Stirling's approximation ($n! \sim \sqrt{2\pi n}(n/e)^n$) to the pre-factor of (6.8) as n tends to infinity (and remembering that $m \sim \xi n/2$) gives

$$\frac{(\xi(\xi+2))^{1/2}4^{1-k}}{\pi n},$$

which implies that (6.8) vanishes in the limit. Hence the correlation function is dominated by the second sum (6.7), chiefly

$$\omega_V(k;\xi) = \lim_{n\to\infty}\Bigg(\frac{2((n+k)/2)!(n-k)!(n-k+1)!}{((n-k)/2)!^2((n-k)/2+2)!^2((n+k)/2-1)!} \times \frac{(2m-1)!(m+n+1)(m+n-1)!(m+n)!}{3(m-2)!(m-1)!(2m+2n-1)!} \times {}_4F_3\left[\begin{matrix}2-k,\ 3/2,\ m+n+2,\ 2-m\\ (n-k+6)/2,\ (n-k+6)/2,\ 5/2\end{matrix};1\right]\Bigg), \quad (6.9)$$

which follows from letting ϵ tend to zero in (6.7).

In order to completely derive the asymptotics of (6.9) it must be shown the limit and sum operations commute in the following expression:

$$\lim_{n\to\infty}\left(\sum_{s=0}^{m}\frac{(2-k)_s(3/2)_s(m+n+2)_s(2-m)_s}{(((n-k+6)/2)_s)^2(5/2)_s(s!)}\right).$$

As this series terminates for $s>k-2$, this is in fact a sum with a finite upper bound dependent only on k and so the limit and the sum in the above expression may be safely interchanged. Doing so, and then applying Stirling's approximation as n tends to infinity, expression (6.9) becomes

$$\omega_V(k;\xi) = \frac{(\xi(\xi+2))^{3/2}4^{1-k}}{3\pi\cdot e}\,{}_2F_1\left[\begin{matrix}2-k,\ 3/2\\ 5/2\end{matrix};-\xi(\xi+2)\right]. \quad (6.10)$$

To uncover the asymptotic behaviour of $\omega_V(k;\xi)$ as k becomes very large it is first convenient to apply transformation (6.3) to the hypergeometric series in (6.10), thereby obtaining

$$\omega_V(k;\xi) = \frac{(\xi(\xi+2))^{3/2}4^{1-k}}{3\pi}\left(\frac{3(\xi+1)^{2k-4}}{2(k-1/2)}\,{}_2F_1\left[\begin{matrix}2-k,\ 1\\ 3/2-k\end{matrix};(\xi+1)^{-2}\right]\right). \quad (6.11)$$

Consider the hypergeometric series in $\omega_V(k;\xi)$ above, that is,

$$\sum_{s=0}^{k-2}\frac{(2-k)_s}{(3/2-k)_s(\xi+1)^{2s}}.$$

It is easy to convince oneself that for any positive even k, $\frac{(2-k)_r}{(3/2-k)_r}\le 1$ for all non-negative s and so

$$\lim_{k\to\infty}\sum_{s=0}^{k-2}\frac{(2-k)_s}{(3/2-k)_s(\xi+1)^{2s}} \le \lim_{k\to\infty}\sum_{s=0}^{k-2}\left(\frac{1}{(\xi+1)^2}\right)^s. \quad (6.12)$$

Since $\xi>0$, the limit on the right hand side of (6.12) exists and is finite so the limit and sum operations on the left side of (6.12) may be interchanged. Hence as k becomes very large,

$$\omega_V(k;\xi) \sim \frac{\sqrt{\xi(\xi+2)}}{2\pi k}\left(\frac{(\xi+1)}{2}\right)^{2(k-1)},$$

which concludes the proof. $\square$

Proof of Theorem 2.3. Consider the correlation function

$$\omega_{H^-}(k;\xi) = \lim_{n\to\infty}\left(\sum_{s=1}^{m} B'_{n,k}(s)\cdot D'_{n,k}(s)\right),$$

where again $m \sim \xi n/2$ and

$$B'_{n,k}(s) = \frac{(-1)^{s+1}(n-k)!(n+s-2)!(n+2s-1)!\left(\frac{1}{2}(k+n+2s-4)\right)!}{2(s-1)!\frac{n-k}{2}!\left(\frac{1}{2}(k+n-2)\right)!(2n+2s-3)!\left(-\frac{k}{2}+\frac{n}{2}+s\right)!},$$

$$D'_{n,k}(s) = \frac{(-1)^{s+1}(2s)!(n-k)!(n+s-1)!\left(\frac{1}{2}(k+n+2s-4)\right)!}{2(s!)\frac{n-k}{2}!\left(\frac{1}{2}(k+n-2)\right)!(n+2s-2)!\left(-\frac{k}{2}+\frac{n}{2}+s\right)!}.$$

The above finite sum may be re-written as a limit of a hypergeometric series:

$$\omega_{H^-}(k;\xi) = \lim_{n\to\infty}\lim_{\epsilon\to 0}\left(\frac{(n-1)!(n+1)!(n-k)!^2}{2(2n-1)!((n-k)/2)!^2((n-k)/2+1)!^2}\right.$$
$$\left.\times {}_7F_6\left[\begin{matrix} n+1+\epsilon,\ \frac{n+3+\epsilon}{2},\ \frac{k+n+\epsilon}{2},\ \frac{k+n+\epsilon}{2},\ \frac{3+\epsilon}{2},\ n+m+1,\ 1-m \\ \frac{n+1+\epsilon}{2},\ \frac{n-k+4+\epsilon}{2},\ \frac{n-k+4+\epsilon}{2},\ \frac{2n+1+\epsilon}{2},\ 1-m+\epsilon,\ m+n+1+\epsilon \end{matrix};1\right]\right).$$

Applying transformation formula (6.2) to the hypergeometric series above and letting ϵ tend to zero gives

$$\omega_{H^-}(k;\xi) = \lim_{n\to\infty}\left(\frac{(2m+1)!((n-k)!)^2(m+n-2)!(m+n)!}{12(m-1)!m!\left(\frac{n-k}{2}!\right)^2\left(\left(\frac{1}{2}(-k+n+2)\right)!\right)^2(2m+2n-3)!}\right.$$
$$\left.\times {}_4F_3\left[\begin{matrix} 2-k,\ 3/2,\ m+n+1,\ 1-m \\ (n-k+4)/2,\ (n-k+4)/2,\ 5/2 \end{matrix};1\right]\right).$$

Consider the limit of this expression as n tends to infinity. By interchanging the sum and limit of the hypergeometric series (as in the previous proof) and applying Stirling's approximation to the pre-factor it may be shown that

$$\omega_{H^-}(k;\xi) = \frac{(\xi(\xi+2))^{3/2}4^{1-k}}{3\pi\cdot e}\,{}_2F_1\left[\begin{matrix} 2-k,\ 3/2 \\ 5/2 \end{matrix};-\xi(\xi+2)\right],$$

which is precisely expression (6.10) above. □

Proof of Theorem 2.1. By the Matchings Factorization Theorem [7] it suffices to consider the asymptotics of the expression

$$\omega_{H^+}(k;\xi) = \lim_{n\to\infty}\left(\sum_{s=1}^{m} B_{n,k}(s)\cdot E_{n,k}(s)\right),$$

where of course $m \sim \xi n/2$, $B_{n,k}(s)$ is as before and

$$E_{n,k}(s) = \frac{(-1)^{s+1}(2s-2)!(-k+n+1)!(n+s-1)!\left(\frac{k}{2}+\frac{n}{2}+s-2\right)!}{(s-1)!\left(\frac{n}{2}-\frac{k}{2}\right)!\left(\frac{k}{2}+\frac{n}{2}-1\right)!(n+2s-2)!\left(-\frac{k}{2}+\frac{n}{2}+s\right)!}.$$

Again this correlation function may be written in as a limit:

$$\omega_{H^+}(k;\xi) = \lim_{n\to\infty}\lim_{\epsilon\to 0}\left(\frac{n!(n+1)!(n-k+1)!^2}{(2n+1)!((n-k)/2)!^2((n-k)/2+1)!^2}\right.$$
$$\left.\times {}_7F_6\left[\begin{matrix} n+1+\epsilon,\ \frac{n+3+\epsilon}{2},\ \frac{k+n+\epsilon}{2},\ \frac{k+n+\epsilon}{2},\ \frac{1+\epsilon}{2},\ n+m+1,\ 1-m \\ \frac{n+1+\epsilon}{2},\ \frac{n-k+4+\epsilon}{2},\ \frac{n-k+4+\epsilon}{2},\ \frac{2n+3+\epsilon}{2},\ 1-m+\epsilon,\ m+n+1+\epsilon \end{matrix};1\right]\right). \tag{6.13}$$

Transforming the above hypergeometric series according to (6.2) and letting ϵ tend to zero gives

$$\omega_{H^+}(k;\xi) = \lim_{n\to\infty}\left(\frac{(2m-1)!(n-k+1)!^2(m+n)!(m+n-1)!}{(m-1)!^2((n-k)/2)!^2((n-k)/2+1)!^2(2m+2n-1)!}\right.$$
$$\left.\times {}_4F_3\left[\begin{matrix} 2-k,\ 1/2,\ m+n+1,\ 1-m \\ (n-k+4)/2,\ (n-k+4)/2,\ 3/2 \end{matrix};1\right]\right) \tag{6.14}$$

As n tends to infinity Stirling's approximation yields

$$\frac{\sqrt{\xi(\xi+2)}}{4^{k-1}\pi\cdot e} \tag{6.15}$$

for the pre-factor while the hypergeometric series reduces to

$${}_2F_1\left[\begin{matrix} 2-k,\ 1/2 \\ 3/2 \end{matrix};-\xi(2+\xi)\right]. \tag{6.16}$$

This follows from interchanging the limit and the summand in (6.14), again by a similar argument to that found in the proof of Theorem 2.2. Applying transformation (6.3) to (6.16), the correlation function $\omega_{H^+}(k;\xi)$ becomes

$$\begin{aligned}\omega_{H^+}(k;\xi) &= \frac{(\xi+1)^{2k-4}}{2(k-3/2)}\,{}_2F_1\left[\begin{matrix} 2-k,\ 1 \\ 5/2-k \end{matrix};(\xi+1)^{-2}\right] \\ &= \frac{1}{2}(\xi+1)^{2k-4}\sum_{s=0}^{k-2}\frac{(k-1-s)_s}{(k-s-3/2)_{s+1}(\xi+1)^{2s}}.\end{aligned}$$

The asymptotic interaction of the holes is given by

$$\lim_{k\to\infty}\omega_{H^+}(k;\xi) = \lim_{k\to\infty}\left(\frac{1}{2}(\xi+1)^{2k-4}\sum_{s=0}^{k-2}\frac{(k-1-s)_s}{(k-s-3/2)_{s+1}(\xi+1)^{2s}}\right). \tag{6.17}$$

It should be clear that

$$\lim_{k\to\infty}\sum_{s=0}^{k-2}\frac{(k-1-s)_s}{(k-s-3/2)_{s+1}(\xi+1)^{2s}} \le \lim_{k\to\infty}\sum_{s=0}^{k-2}\left(\frac{1}{(\xi+1)^2}\right)^s,$$

so since the limit on the right exists and is finite the sum and limit operations in (6.17) may be interchanged. Combining (6.17) with (6.15) and letting k become very large gives

$$\omega_{H^+}(k;\xi) \sim \left(\frac{(\xi+1)}{2}\right)^{2k-2}\frac{1}{2\pi k\sqrt{\xi(\xi+2)}}.$$

Inserting this into Proposition 2.4 along with the result from Theorem 2.3 yields

$$\omega_H(k;\xi) \sim \left(\frac{1}{2k\pi} \left(\frac{\xi+1}{2} \right)^{2k-2} \right)^2 .$$

This completes the proof. $\square$

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